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Axiomatic characterization of nonlinear homomorphic means

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1. Introduction

One important problem in mathematics and statistics is to formulate a mean from a set of data. Various definitions of mean have been studied to represent in some sense an average of data. The applications of these means involve all the fields of science and technology, with nonlinear filtering being a primary example in the area of image processing [1–3]. The goal of this work is to axiomatically investigate the rationale for a large class of means, which is referred to as *quasi-arithmetic mean* [4], *nonlinear homomorphic mean* or *nonlinear homomorphic filter* [3].

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Let $x = (x_1, x_2, \dots, x_n)$ denote a set of data and $F(x)$ a mean of these data. $F(x)$ is called a nonlinear homomorphic mean (or filter in the field of area processing) of $x = (x_1, x_2, \dots, x_n)$, if

$$F(x) = U^{-1}\left[\sum_{i=1}^{n} p_i U(x_i)\right], \tag{1}$$

where U is a differentiable and strictly increasing function, $\{p_i\}_{i=1}^n$ are nonnegative weighting constants such that $\sum_{i=1}^n p_i = 1$. In the image processing literature, nonlinear homomorphic filters are successfully used for noise reduction [1–3].

There are several axiomatic studies characterizing the mean as the homomorphic filter given in (1). The axiomatic approach was first used by Kolmogorov and Nagumo respectively [5,6], in the case of $p_i = 1/n$ and U being continuous (not necessarily differentiable). The key axiom they proposed is a *recursive associative* functional equation that the mean function sequence $F_n(x_1, \dots, x_n)$ for $n \geqslant k \geqslant 1$ should satisfy[3]

$$F_n(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = F_n(F_k, \dots, F_k, x_{k+1}, \dots, x_n), \tag{2}$$

where F_k stands for $F_k(x_1, \dots, x_k)$. Their other requirements include the continuity, symmetry, monotonic increment, and idempotence axioms [5,6] (see Section 2). This work was extended to cover the weighted mean [7,8]. Also, the recursive associative property was replaced by a *bisymmetric* condition in [9,10].

The mean can also be viewed as a function for synthesizing judgements. If numerical judgements $x_1, \dots, x_n$ are given in a positive interval P, the synthesizing function F maps P^n into a proper interval J [4]. The following *separability condition* is the key axiom for this approach:

$$F(x_1, \dots, x_n) = g(x_1) \circ \cdots \circ g(x_n), \quad \forall x_1, \dots, x_n \in P, \tag{3}$$

where g is a function mapping P onto a proper interval J, and $\circ$ is a continuous, associative and cancellative mapping from $J \times J$ into J [4,11,12]. It is known that those conditions about the mapping $\circ$ are enough to guarantee that there exists a continuous and strictly monotonic function Ψ, such that [13]

$$u \circ v = \Psi^{-1}\big(\Psi(u) + \Psi(v)\big). \tag{4}$$

If F further satisfies the unanimity condition (i.e., the idempotence axiom), then $\Psi(x) = n\Psi[g(x)]$. Under these conditions, F is of the form (1) with the weighting constants $p_i = 1/n$ and U being continuous [4,12].

Given the importance of the homomorphic mean and its applications, from a new perspective we report an axiomatic characterization of the mean as follows. The key axiom is the following perturbation equation:

$$V\big[F(x), F(x + \Delta x) - F(x)\big] = \sum_{i=1}^{n} p_i V(x_i, \Delta x_i) + o\big(\|\Delta x\|\big),$$

[3] Here the mean function is denoted by $F_n(x)$ to emphasize its dependency on n. However, we will generally use the notation $F(x)$ without loss of clarity.

when $\|\Delta x\|$ is sufficiently small, where $\|\cdot\|$ is the canonical Euclidean norm, p_i are positive constants with $\sum_{i=1}^{n} p_i = 1$, and the function $V(\cdot,\cdot)$ is a measure on variation of a quantity (either data or mean). More precisely, $V(\alpha, \beta)$ is a measure of the variation from α to $\alpha + \beta$. The same measure is applied upon both perturbation in data and change in the corresponding mean. This axiom simply requires that the perturbation in mean is linearly dependent on the variation of each of the datum. The condition on the constants p_i further restricts that the mean perturbation be contained in the convex polyhedron with the quantity variations as the vertices.

Despite the mathematical elegance of the previous studies [4–10,12], which are based on functional equations, our approach is easier to be interpreted in practical applications. With reasonable quantity variation functions, a large class of means can be induced, which also includes new means (see Section 4). It is proved in Theorems 3.1 and 3.2 that the induced nonlinear homomorphic mean is uniquely determined by $\partial V(\alpha, 0)/\partial\beta$. Conversely, Theorem 3.3 shows that each nonlinear homomorphic mean F in (1) can be generated by a quantity variation function V as in Theorems 3.1 and 3.2. Although it may happen that different choices of the quantity variation function V can induce the same mean function (see Section 4), the relationship as characterized in Theorems 3.1–3.3 provides the guideline for selection of the quantity variation function. Note that the perturbation axiom is well motivated, because it is quite reasonable to measure the change of a quantity, hence to speculate a fundamental link among such changes in data and their mean.

The structure of this paper is as follows. In Section 2, notations and axioms are presented. The axioms include smoothness, symmetry, idempotence, translation, scaling, and perturbation axioms. In Section 3, the main results are proved, including Theorems 3.1–3.3. In Section 4, classic log–exp, arithmetic, rth power and geometric means are rediscovered by appropriately setting the quantity variation function V, and new means are derived to demonstrate the fertility of the perturbation axiom. In Section 5, those classic means are further characterized from the perspective of different invariant properties, respectively.

2. Notations and axioms

The domain of the mean function is the positive space $\mathbf{R}_+^n = \{x \in \mathbf{R}^n\colon\ x_i > 0,\ i = 1, \ldots, n\}$. For any $t \in \mathbf{R}$, let $\vec{t}$ be the vector in $\mathbf{R}^n$, with each of its components equal to t, i.e., $\vec{t} = (t, \ldots, t) \in \mathbf{R}^n$. We postulate the following axioms.

Axiom 1 (Smoothness). F is continuous, and $\partial F/\partial x_i$ exists, $i = 1, \ldots, n$.

Axiom 2 (Symmetry). $F(x_1, x_2, \ldots, x_n) = F(x_1', x_2', \ldots, x_n')$, where $x_1', x_2', \ldots, x_n'$ is any permutation of $x_1, x_2, \ldots, x_n$.

Axiom 3 (Idempotence). $F(\vec{c}\,) = c$ for any $c > 0$.

Axiom 4 (Translation). $F(x + \vec{t}\,) = F(x) + t$ for any $t > 0$.

Axiom 5 (Scaling). $F(sx) = sF(x)$ for any $s > 0$.

Axiom 6 (Perturbation). $V[F(x), F(x+\Delta x) - F(x)] = \sum_{i=1}^{n} p_i V(x_i, \Delta x_i) + o(\|\Delta x\|)$, where $V(\alpha, \beta)$ quantifies the variation from α to $\alpha + \beta$ with $V(\alpha, 0) = 0$, $\nu(\alpha, \beta) \triangleq \partial V(\alpha, \beta)/\partial \beta > 0$ is continuous with respect to α, β,[4] and p_i are positive constants with $\sum_{i=1}^{n} p_i = 1$.

Remark 2.1. The domain of the mean function is restricted to $\mathbf{R}_+^n$ because a quantity variation function can be readily appreciated in the positive space. When the mean function is determined in $\mathbf{R}_+^n$, it is often feasible to extend it to the whole space $\mathbf{R}^n$ (see the examples in Section 4).

The smoothness axiom excludes the possibility that a small modification on a datum would cause a dramatic change in the mean. Any mean function that violates the smoothness axiom should not be considered. The symmetry axiom implies that a mean should be a collective property regardless of the indexing of the data. By the nature of the mean, the idempotence axiom is self-evident. The translation and scaling axioms describe invariant properties under translation and scaling transforms, respectively.

While all the other axioms have already been mentioned in the literature, we underline that the perturbation axiom is our contribution for the axiomatic mean theory. For characterization of nonlinear homomorphic means, only the smoothness, idempotence and perturbation axioms are needed. Other axioms further restrict the homomorphic mean function to more concrete forms (see Section 5).

3. Main results

Theorem 3.1. *Assume that F is a function satisfying the smoothness, idempotence and perturbation axioms, F is of the form*

$$F(x) = U^{-1}\left[\sum_{i=1}^{n} p_i U(x_i)\right], \tag{5}$$

where U is differentiable and strictly increasing, $\{p_i\}_{i=1}^{n}$ are the constants given in the perturbation axiom.

Proof. By the Taylor expansion with F and V, we have

$$V\big[F(x), F(x+\Delta x) - F(x)\big] = V\left[F(x), \sum_{i=1}^{n} \frac{\partial F(x)}{\partial x_i}\Delta x_i + o\big(\|\Delta x\|\big)\right] \tag{6}$$

$$= V\big[F(x), 0\big] + \nu\big[F(x), 0\big]\left[\sum_{i=1}^{n} \frac{\partial F(x)}{\partial x_i}\Delta x_i\right] + o\big(\|\Delta x\|\big). \tag{7}$$

By the perturbation axiom,

[4] In fact, we only need to assume that $\nu(\alpha, 0)$ is continuous with respect to $\alpha > 0$.

$$\nu\big[F(x),0\big]\Bigg[\sum_{i=1}^{n}\frac{\partial F(x)}{\partial x_i}\Delta x_i\Bigg]=\sum_{i=1}^{n}p_i V(x_i,\Delta x_i)+o\big(\|\Delta x\|\big) \tag{8}$$

$$=\sum_{i=1}^{n}p_i\nu(x_i,0)\Delta x_i+o\big(\|\Delta x\|\big). \tag{9}$$

Therefore,

$$\nu\big[F(x),0\big]\frac{\partial F(x)}{\partial x_i}=p_i\nu(x_i,0) \tag{10}$$

for each x_i, $i=1,\dots,n$. Let

$$U(\eta)\triangleq\int_{\eta_0}^{\eta}\nu(\alpha,0)\,d\alpha, \tag{11}$$

where η_0 is a constant such that the above integral exists for all $\eta>0$.[5] Then, (10) can be written as

$$\frac{\partial}{\partial x_i}[U\circ F](x)=p_iU'(x_i) \tag{12}$$

for each x_i, $i=1,\dots,n$. Let $\vec{1}$ be the vector of $\mathbf{R}_+^n$ with all the components being 1. Then we have

$$\begin{aligned}
U\big[F(x)\big]-U\big[F(\vec{1})\big]&=U\big[F\big(\vec{1}+t(x-\vec{1})\big)\big]\Big|_{t=0}^{t=1}\\
&=\int_0^1\frac{d}{dt}U\big[F\big(\vec{1}+t(x-\vec{1})\big)\big]\,dt\\
&=\int_0^1\sum_{i=1}^{n}\frac{\partial}{\partial x_i}[U\circ F]\big(\vec{1}+t(x-\vec{1})\big)\cdot(x_i-1)\,dt\\
&=\int_0^1\sum_{i=1}^{n}p_iU'\big(1+t(x_i-1)\big)\cdot(x_i-1)\,dt\\
&=\sum_{i=1}^{n}p_iU\big(1+t(x_i-1)\big)\Big|_{t=0}^{t=1}=\sum_{i=1}^{n}p_iU(x_i)-\sum_{i=1}^{n}p_iU(1)\\
&=\sum_{i=1}^{n}p_iU(x_i)-U(1),
\end{aligned}$$

[5] The constant η_0 may be a positive finite constant or $+\infty$ (see Section 4).

where we have used the assumptions that $\sum_{i=1}^{n} p_i = 1$ in the perturbation axiom. By the idempotence axiom, $F(\vec{1}) = 1$. Hence, $U[F(\vec{1})] = U(1)$. Therefore, it follows that

$$U\big[F(x)\big] = \sum_{i=1}^{n} p_i U(x_i). \tag{13}$$

Note that $U'(\eta) > 0$ by the definitions of the U and V functions. The proof follows immediately from (13). □

Incorporating the symmetry axiom, we can easily prove that

Corollary 3.2. *Under the assumptions of Theorem* 3.1, *if F satisfies further the symmetry axiom, F is of the form*

$$F(x) = U^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} U(x_i)\right], \tag{14}$$

where U is the same as in Theorem 3.1.

Theorem 3.3. *Assume that F_* is a nonlinear homomorphic mean as in* (1). *If $U \in C^1(\mathbf{R}_+)$, then there is a quantity variation function V_* such that the mean function F as determined in Theorem* 3.1 *is equal to F_*.*

Proof. The proof is based on the following construction. Let

$$\nu_*(\alpha, \beta) = U'(\alpha), \tag{15}$$

$$V_*(\alpha, \beta) = U'(\alpha)\beta. \tag{16}$$

By the proof of Theorem 3.1, the mean function F as determined in Theorem 3.1 by the quantity variation function V_* is given by

$$F(x) = U^{-1}\left[\sum_{i=1}^{n} p_i U(x_i)\right].$$

The result follows immediately. □

4. Means induced by quantity variation functions

In this section, a number of classic homomorphic means are induced by appropriate choices of the quantity variation function V. Furthermore, new homomorphic means are also formulated by some fairly simple quantity variation functions.

Weighted arithmetic mean. When the quantity variation is measured by the *absolute error*, we have $V(\alpha, \beta) = \beta$. Then,

$$\nu(\alpha, 0) = 1, \tag{17}$$

$$U(\eta) = \eta, \quad \text{with } \eta_0 = 0, \tag{18}$$

$$F(x) = \sum_{i=1}^{n} p_i x_i. \tag{19}$$

Weighted geometric mean. When the quantity variation is measured by the *relative error*, we have $V(\alpha, \beta) = \beta/\alpha$. Then,

$$\nu(\alpha, 0) = \frac{1}{\alpha}, \tag{20}$$

$$U(\eta) = \log \eta, \quad \text{with } \eta_0 = 1, \tag{21}$$

$$F(x) = \prod_{i=1}^{n} x_i^{p_i}. \tag{22}$$

Weighted harmonic mean. Let $V(\alpha, \beta) = \beta/\alpha^2$, which is a relative measure of the relative error and may be more appropriate than the classic absolute and relative errors in some special situations. Then,

$$\nu(\alpha, 0) = \frac{1}{\alpha^2}, \tag{23}$$

$$U(\eta) = -\frac{1}{\eta}, \quad \text{with } \eta_0 = \infty, \tag{24}$$

$$F(x) = \frac{1}{\sum_{i=1}^{n} (p_i/x_i)}. \tag{25}$$

***p*th weighted mean.** Let $V(\alpha, \beta) = \beta\alpha^{p-1}$ for some $p > 0$. Then,

$$\nu(\alpha, 0) = \alpha^{p-1}, \tag{26}$$

$$U(\eta) = \frac{1}{p}\eta^p, \quad \text{with } \eta_0 = 0, \tag{27}$$

$$F(x) = \left(\sum_{i=1}^{n} p_i x_i^p\right)^{1/p}. \tag{28}$$

Negative *p*th weighted mean. Let $V(\alpha, \beta) = \beta/\alpha^{1+p}$ for some $p > 0$. Then,

$$\nu(\alpha, 0) = \frac{1}{\alpha^{1+p}}, \tag{29}$$

$$U(\eta) = -\frac{1}{p\eta^p}, \quad \text{with } \eta_0 = \infty, \tag{30}$$

$$F(x) = \left(\sum_{i=1}^{n} p_i x_i^{-p}\right)^{-1/p}. \tag{31}$$

Weighted log–exp mean. Let $V(\alpha,\beta)=\beta c^{\alpha}$ for some $c>0$. Then,

$$\nu(\alpha,0)=c^{\alpha}, \tag{32}$$

$$U(\eta)=\frac{c^{\eta}-1}{\log c}, \quad \text{with } \eta_0=0, \tag{33}$$

$$F(x)=\log_c\left(\sum_{i=1}^{n} p_i c^{x_i}\right). \tag{34}$$

Let $a=\log c$. Then, the log–exp mean is obtained as

$$F(x)=\frac{1}{a}\log\sum_{i=1}^{n} p_i \mathbf{e}^{a x_i}. \tag{35}$$

Note that the following two homomorphic means are new, although similar means involving the arctan function can be found in [14, Theorem 10]. They show a new way to construct and study mean functions. Also, it would be inspiring and could be fruitful to explore further along this direction.

Weighted arctan–tan mean. Let $V(\alpha,\beta)=\beta/(1+\alpha^2)$. Then,

$$\nu(\alpha,0)=\frac{1}{1+\alpha^2}, \tag{36}$$

$$U(\eta)=\arctan\eta, \quad \text{with } \eta_0=0, \tag{37}$$

$$F(x)=\tan\left(\sum_{i=1}^{n} p_i \arctan x_i\right). \tag{38}$$

Weighted shifted arctan–tan mean. Let $V(\alpha,\beta)=\beta/(1+\alpha+\alpha^2)$. Then,

$$\nu(\alpha,0)=\frac{1}{1+\alpha+\alpha^2}, \tag{39}$$

$$U(\eta)=\frac{-\pi}{3\sqrt{3}}+\frac{2\arctan(\frac{1+2x}{\sqrt{3}})}{\sqrt{3}}, \quad \text{with } \eta_0=0, \tag{40}$$

$$F(x)=\frac{\sqrt{3}\tan(\sum_{i=1}^{n} p_i \arctan(\frac{1+2x_i}{\sqrt{3}}))-1}{2}. \tag{41}$$

Weighted shifted harmonic mean. Let $V(\alpha,\beta)=\beta/(1+2\alpha+\alpha^2)$. Then,

$$\nu(\alpha,0)=\frac{1}{1+2\alpha+\alpha^2}, \tag{42}$$

$$U(\eta)=1-\frac{1}{1+\eta}, \quad \text{with } \eta_0=0, \tag{43}$$

$$F(x)=1-\frac{1}{\sum_{i=1}^{n}(p_i/(x_i+1))}. \tag{44}$$

Remark 4.1. The quantity variation can also be measured in a log-scale of the original data. It is found that the same homomorphic means are induced by different choices of the quantity variation function. We just list the results as follows:

$$V(\alpha,\beta)=\log\frac{\alpha+\beta}{\alpha},\quad U(\eta)=\log\eta\quad\Rightarrow\quad\text{weighted geometric mean;}$$
$$V(\alpha,\beta)=(\alpha+\beta)\log\frac{\alpha+\beta}{\alpha},\quad U(\eta)=\eta\quad\Rightarrow\quad\text{weighted arithmetic mean;}$$
$$V(\alpha,\beta)=\alpha\log\frac{\alpha+\beta}{\alpha},\quad U(\eta)=\eta\quad\Rightarrow\quad\text{weighted arithmetic mean.}$$

5. Means characterized by axioms

The previous section shows that a large class of homomorphic means can be induced by quantity variation functions V in the perturbation axiom. In this section, several classic means are characterized by combinations of the invariance axioms in Section 2 under a higher order smoothness assumption on the quantity variation function V.

Theorem 5.1. *Under the assumptions of Theorem* 3.1, *assume the quantity variation function V is C^2.*[6] *If F additionally satisfies the translation axiom, F is one of the following two classic homomorphic means*:

(1) *the weighted* log–exp *mean in* (35) *for some constant $a \neq 0$, or*
(2) *the weighted arithmetic mean in* (19).

Note that the weighted arithmetic mean is the limit of the log–exp mean as $a \to 0$.

Proof. We only need to prove the theorem for $n \geqslant 2$. By Theorem 3.1, for any $x \in \mathbf{R}_+^n$,

$$U\big(F(x)\big)=\sum_{i=1}^{n}p_iU(x_i). \tag{45}$$

Taking the derivative with respect to x_k, for any $1 \leqslant k \leqslant n$,

$$U'\big(F(x)\big)\frac{\partial F(x)}{\partial x_k}=p_kU'(x_k). \tag{46}$$

By the translation axiom, for any $t > 0$,

$$U\big(F(x)+t\big)=U\big(F(x+\vec{t}\,)\big)=\sum_{i=1}^{n}p_iU(x_i+t).$$

[6] In fact, we only need to assume that $\nu(\alpha,0)$ in the perturbation axiom is C^1 as a function of α.

By the assumption on the function V, U as constructed in (11) is C^2. Taking the derivative on both sides of the above formula with respect to t and then letting $t \to 0$, we have

$$U'\big(F(x)\big) = \sum_{i=1}^{n} p_i U'(x_i). \tag{47}$$

For any $1 \leqslant k \leqslant n$, taking the derivative with respect to x_k, we have

$$U''\big(F(x)\big)\frac{\partial F(x)}{\partial x_k} = p_k U''(x_k). \tag{48}$$

By (46) and the above equation, we have, noting that $U'(t) \neq 0$ for any $t > 0$ by its construction and the assumption in the perturbation axiom,

$$\frac{U''(F(x))}{U'(F(x))} = \frac{U''(x_k)}{U'(x_k)} \tag{49}$$

for any $1 \leqslant k \leqslant n$. Hence, it follows that

$$\frac{U''(x_j)}{U'(x_j)} = \frac{U''(x_k)}{U'(x_k)} \tag{50}$$

for any $1 \leqslant j, k \leqslant n$ and that this ratio is a constant. Let it be denoted by a. Then, we have

$$\frac{U''(t)}{U'(t)} = a \tag{51}$$

for $t > 0$. That is,

$$U''(t) - aU'(t) = 0, \tag{52}$$

or

$$\big(\mathbf{e}^{-at}U'(t)\big)' = 0. \tag{53}$$

Hence, there is a constant b such that

$$U'(t) = b\mathbf{e}^{at}, \tag{54}$$

where $b \neq 0$, since $U'(t) \neq 0$.

We consider the following two cases.

Case I. $a \neq 0$. By (47),

$$b\mathbf{e}^{aF(x)} = b\sum_{i=1}^{n} p_i \mathbf{e}^{ax_i}. \tag{55}$$

Hence,

$$F(x) = \frac{1}{a}\log\sum_{i=1}^{n} p_i \mathbf{e}^{ax_i} \tag{56}$$

is the weighted log–exp mean in (35).

Case II. $a = 0$. By (54), $U'(t) = b$, a constant. Hence, $U(t) = bt + c$, for some constant c. By (45),

$$bF(x) + c = b\sum_{i=1}^{n} p_i U(x_i) + c.$$

Hence, it follows immediately that

$$F(x) = \sum_{i=1}^{n} p_i U(x_i)$$

is the weighted arithmetic mean in (19). □

Corollary 5.2. *Under the assumptions in Theorem* 5.1. *If F also satisfies the scaling axiom, F is the arithmetic mean as in* (19).

Proof. Since the log–exp mean does not satisfy the scaling axiom, the result follows easily from Theorem 5.1. □

In fact, the arithmetic mean can be characterized without the perturbation axiom. Specifically, using the smoothness, idempotence, translation and scaling axioms, we can establish the arithmetic mean as follows.

Theorem 5.3. *To be consistent with the smoothness, translation and scaling axioms, $F(x)$ is the weighted arithmetic mean as in* (19)*. If $F(x)$ is further restricted by the symmetry axiom, it is the arithmetic mean*

$$F(x) = \frac{1}{n}\sum_{i=1}^{n} x_i. \tag{57}$$

Proof. Let $g_k(x) = \partial F(x)/\partial x_k$ for $1 \leqslant k \leqslant n$. By the translation axiom and after differentiation with respect to x_k, we have

$$g_k(x + \vec{t}\,) = g_k(x), \quad \forall t > 0. \tag{58}$$

Similarly by the scaling axiom, we have

$$g_k(sx) = g_k(x), \quad \forall s > 0. \tag{59}$$

By (59), we have

$$g_k(x) = g_k\left(\frac{x}{\|x\|}\right), \quad \forall x \in \mathbf{R}_+^n. \tag{60}$$

Then, by (58),

$$g_k(x) = g_k\left(\frac{x + \vec{t}}{\|x + \vec{t}\,\|}\right), \quad \forall x \in \mathbf{R}_+^n,\ \forall t > 0. \tag{61}$$

As $t \to +\infty$, $(x + \vec{t}\,)/\|x + \vec{t}\,\| \to \vec{1}$. Letting $t \to \infty$, we have

$$g_k(x) = g_k(\vec{1}), \quad \forall x \in \mathbf{R}_+^n. \tag{62}$$

Hence, g_k is a constant valued function. Let $g_k(x) = p_k$ for $1 \leqslant k \leqslant n$. By the translation axiom and after partial differentiation with respect to t, we have that

$$\sum_{k=1}^{n} g_k(x) = 1. \tag{63}$$

Therefore,

$$\sum_{k=1}^{n} p_k = 1. \tag{64}$$

Finally, by the scaling axiom and after partial differentiation with respect to s, we have

$$\sum_{k=1}^{n} x_k g_k(sx) = F(x). \tag{65}$$

Therefore, $F(x)$ is the weighted arithmetic mean

$$F(x) = \sum_{k=1}^{n} p_k x_k. \tag{66}$$

If $F(x)$ is additionally requested to satisfy the symmetry axiom, it is easy to prove that all the p_k are equal to $1/n$ as in the proof of Theorem 3.2. □

Moreover, using the smoothness, idempotence, scaling and perturbation axioms, we can characterize either the rth weighted power mean or the weighted geometric mean.

Theorem 5.4. *Under the assumptions of Theorem* 3.1*, and assume that the quantity variation function V in the perturbation axiom is C^2. If F additionally satisfies the scaling axiom, it is one of the following two homomorphic means*:

(1) *the pth weighted power mean in* (28) *for some constant $p \neq 0$, or*
(2) *the weighted geometric mean in* (22).

Note that the weighted geometric mean is the limit of the pth weighted power mean as $r \to 0$.

Proof. We only need to prove the theorem for $n \geqslant 2$. By Theorem 3.1 and the scaling axiom, for any $x \in \mathbf{R}_+^n$ and $s > 0$,

$$U\big(sF(x)\big) = U\big(F(sx)\big) = \sum_{i=1}^{n} p_i U(sx_i).$$

By the assumption on the function V, U as constructed in (11) is C^2. Taking the derivative on both sides of the above formula with respect to s and letting $s \to 1$, we have

$$U'\big(F(x)\big)F(x) = \sum_{i=1}^{n} p_i U'(x_i)x_i. \tag{67}$$

For any $1 \leqslant k \leqslant n$, taking the derivative with respect to x_k in (67), we have

$$\left[U''\left(F(x)\right)F(x) + U'\left(F(x)\right)\right]\frac{\partial F(x)}{\partial x_k} = p_k\left(U''(x_k)x_k + U'(x_k)\right).$$

Similar to what has been done in the proof of Theorem 5.1, by (46) and the above equation, we find that

$$\frac{U''(F(x))F(x) + U'(F(x))}{U'(F(x))} = \frac{U''(x_k)x_k + U'(x_k)}{U'(x_k)}, \tag{68}$$

and that this ratio is a constant. Let it be denoted by p, we have

$$\frac{U''(t)t + U'(t)}{U'(t)} = p \tag{69}$$

for $t > 0$. Then,

$$U''(t) - \frac{\alpha}{t}U'(t) = 0, \tag{70}$$

where $\alpha = p - 1$. Then,

$$\left(\mathbf{e}^{-\alpha \log t} U'(t)\right)' = 0. \tag{71}$$

Hence, there is a constant b such that

$$U'(t) = b\mathbf{e}^{\alpha \log t} = bt^{\alpha}. \tag{72}$$

$b \neq 0$ since $U'(t) \neq 0$.

We consider the following two cases.

Case I. $\alpha \neq -1$. By (67),

$$bF(x)^{\alpha}F(x) = b\sum_{i=1}^{n} x_j^{\alpha}x_j. \tag{73}$$

Hence, it follows immediately that $F(x)$ is the pth weighted mean (28).

Case II. $\alpha = -1$. By (72), $U'(t) = b/t$. Hence, $U(t) = b\log t + c$, for some constant c. By (45),

$$b\log F(x) + c = b\sum_{i=1}^{n} p_i \log x_i + c.$$

Hence, it follows immediately that $F(x)$ is the weighted geometric mean in (22). □

References

[1] A. Kundu, S.K. Mitra, P.P. Vaidyanathan, Generalized mean filters: A new class of nonlinear filters for image processing, in: Proc. 6th Sympos. Circuit Theory Design, 1983, pp. 185–187.

[2] I. Pita, A.N. Venetsanopoulos, Nonlinear mean filters in image processing, IEEE Trans. Acoust. Speech Signal Process. 34 (1986) 573–584.

[3] I. Pita, A.N. Venetsanopoulos, Nonlinear Digital Filters: Principles and Applications, Kluwer Academic, Boston, 1990.
[4] J. Aczél, C. Alsina, Synthesizing judgments: a functional equations approach, Math. Modelling 9 (1987) 311–320.
[5] A.N. Kolmogoroff, Sur la notion de la moyenne, Atti Reale Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. 12 (1930) 388–391.
[6] M. Nagumo, Über eine Klasse der Mittelwerte, Japanese J. Math. 7 (1930) 71–79.
[7] B. de Finetti, Sul concello di media, Giornale Istit. Italiano Attuarii 2 (1931) 311–320.
[8] T. Kitagawa, On some class of weighted means, Proc. Phys.-Math. Soc. Japan 16 (1934) 311–320.
[9] J. Aczél, On mean values, Bull. Amer. Math. Soc. 54 (1948) 392–400.
[10] J.C. Fodor, J.L. Marichal, On nonstrict means, Aequationes Math. 54 (1997) 308–327.
[11] T.L. Saaty, The Analytic Hierarchy Process, McGraw–Hill, New York, 1980.
[12] J. Aczél, On weighted synthesis of judgements, Aequationes Math. 27 (1984) 288–307.
[13] J. Aczél, Lectures on Functional Equations and Their Applications, Academic Press, New York, 1966.
[14] J. Matkowski, Mean value property and associated functional equations, Aequationes Math. 58 (1999) 46–59.