Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 315 (2006) 462–473
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

A general axiomatic system for image resolution quantification

Ming Jiang [a,∗,1], Ge Wang [b,2], Xiao-ming Ma [c]

[a] *LMAM, School of Mathematical Sciences, Peking University, 5 Summer Palace Street, Beijing 100871, China*
[b] *CT/Micro-CT Laboratory, Departments of Radiology, Biomedical Engineering, and Mathematics, University of Iowa, Iowa City, IA 52242, USA*
[c] *Center of Environment Science, Peking University, Beijing 100871, China*

Received 5 June 2004
Available online 17 November 2005
Submitted by U. Stadtmueller

Abstract

In this paper, we generalize our previously published axiom system for quantification of image resolution and prove that any resolution measure consistent with the new axiom system must be a homogeneous symmetric function of order $1/2$ of the eigenvalues of the covariance matrix of the PSF. We demonstrate that the previous axiom system is not consistent with the affine transformation axiom. We propose a weak combination axiom to replace the previous strong combination axiom and use it to solve this conflict. It is remarkable that the original finding in one-dimension by Wang and Li can be easily rediscovered with aid of the weak combination axiom, instead of using the previous strong combination axiom. If the previous axiom system is modified with the weak combination axiom and augmented with the affine transformation axiom, the resolution measure is shown to be proportional to the squared root of the geometric mean of the eigenvalues of the covariance matrix of the PSF. Relevant discussions and possible extensions are also provided.

Keywords: Imaging system; Point spread function (PSF); Image resolution; Arithmetic mean; Geometric mean

∗ Corresponding author. Fax: +86 10 62751801.
E-mail addresses: ming-jiang@ieee.org, jiangm@math.pku.edu.cn (M. Jiang), ge-wang@ieee.org (G. Wang), xmma@ces.pku.edu.cn (X. Ma).
[1] Partially supported by the National Basic Research Program of China under Grant 2003CB716101, National Science Foundation of China under Grants 60325101, 60272018 and 60372024, and Engineering Research Institute, Peking University.
[2] Partially supported by the NIH/NIBIB under grants DC03590, EB002667, EB001685, USA.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.10.039

1. Introduction

It is a long-standing fundamental issue in imaging how to quantify the image resolution of an imaging system. There are various kinds of imaging systems, such as cameras, electronic and optic microscopes, medical and industrial tomographic scanners, radars, etc. Image resolution of an imaging system refers to the capability of discriminating two neighboring small objects, and is the primary factor characterizing the system performance. Over the past years, various criteria have been introduced for the quantification of image resolution including Rayleigh's criterion, bandwidth of the modular transfer function (MTF), cut-off frequency of the MTF, full-width-at-half-maximum (FWHM), full-width-at-tenth-maximum (FWTM), area/volume under the MTF curve/surface (i.e., the Strehl ratio), and standard deviation of the point spread function (PSF) [1]. Image resolution is extremely useful, because "*it is desirable to have some simple criterion which permits a rough comparison of the relative efficiency of different systems*" [2, p. 461, §8.6.2]. It is remarkable to note that this issue can be addressed using an axiomatic approach [3,4]. Here we extend this method to general cases and deduce a new image resolution measure.

The following model is widely accepted for many imaging systems:

$$g = p * f, \tag{1}$$

where f is the input or original image, g the output or observed image or data, and p the point spread function (PSF) of the imaging system. The above convolution model is a spatially invariant linear system. The performance of the imaging system is completely determined by its PSF p. Although the model (1) is the simplest in the imaging field, it represents a powerful approximation to a number of important practical systems, with X-ray computed tomography (CT) as a primary example.

Spatial resolution of X-ray CT images depends upon imaging geometry and reconstruction strategy [5]. To obtain high resolution images, the direct approach is to perform image reconstruction from projection data typically using an advanced algorithm [6]. An indirect approach is to apply an image restoration technique to process reconstructed images based on an appropriate distortion model. One model is the linear spatially invariant distortion process as in (1), which represents a very good approximation in many cases such as when the region of interest is not too large. For sequential CT it was found that the PSF p can be well fitted by a Gaussian PSF and identified in a phantom experiment [7,8]. This model was applied for CT image enhancement [9–12]. Currently, spiral CT is one of the most popular medical imaging modalities. In our previous work, the same linear spatially invariant model (1) was validated in the cases of single- and multi-slice CT [13,14], and applied for deblurring and blind deblurring of CT images [14,15]. In [16], this model was based upon to study spatial variation in image resolution of spiral CT in terms of our previously characterized axiomatic resolution measure [3,4]. Similar comments can be made on PET (positron emission tomography) and SPECT (single photon emission computed tomography), two other important medical imaging modalities that are linear systems [2,13–15,17].

In this work, let $R[\cdot]$ denote an image resolution measure. Then, $R[p]$ is the image resolution for the system (1). The heuristics is that the smaller $R[p]$ is, the finer details the system can resolve. To define the class of imaging systems under study precisely, we need the following definitions and notations. For a PSF p, its mean vector $\mu[p] = (\mu_1[p], \dots, \mu_N[p])$ is defined by

$$\mu_k[p] = \frac{\int_{\mathbf{R}^N} x_k \cdot p(x)\,dx}{\int_{\mathbf{R}^N} p(x)\,dx}, \quad k = 1, \dots, N, \tag{2}$$

and its covariance matrix $\sigma[p]$ by

$$\sigma[p]_{j,k} = \frac{\int_{\mathbf{R}^N} (x_j - \mu_j[p])(x_k - \mu_k[p]) \cdot p(x)\,dx}{\int_{\mathbf{R}^N} p(x)\,dx}, \quad j = 1, \ldots, N,\ k = 1, \ldots, N, \tag{3}$$

which are the first two moments of p. We assume that all the integrals involved exist and ratios are well defined in the above formulas. The class of imaging systems under study is specified by the following admissible PSF set:

$$\mathcal{P} = \bigl\{p \in L^1\bigl(\mathbf{R}^N\bigr):\ \mu[p] \text{ and } \sigma[p] \text{ are finite},\ \sigma[p] \text{ is positive definite}\bigr\}. \tag{4}$$

For a nonnegative $p \in \mathcal{P}$, $\sigma[p]$ is positive definite. For example, this holds for the cases of X-ray CT, PET and SPECT. When $N = 1$, the covariance matrix reduces to the conventional variance.

For $N = 1$, it was proved in [3] that $R[p]$ is proportional to the squared root of the variance of p under certain axiomatic assumptions about $R[p]$. This axiomatic approach was extended to the case of $N > 1$ in [4]. It was proved in [4] that $R[p]$ is proportional to the square root of the arithmetic mean of the eigenvalues of the covariance matrix of the PSF p in an augmented axiomatic system. The axioms proposed in [4] was a natural multi-dimensional extension of those in [3]. Many of them are so general that we still need them in the present work. In the following, we first introduce those axioms and represent the main result of [4]. Then we present our extensions.

Let I denote the identity matrix of any order. For an $N \times N$ positive definite matrix σ, $N \geqslant 1$, G^σ denotes the following Gaussian PSF:

$$G^\sigma(x) = \exp\left(-\frac{1}{2} x^{\mathrm{tr}} \cdot \sigma^{-1} \cdot x\right), \quad \text{for } x \in \mathbf{R}^N, \tag{5}$$

with the covariance matrix σ. Note that G^σ is in the admissible PSF set $\mathcal{P}$. The axioms previously proposed in [4] are as follows.

Axiom 1 (*Nonnegativity*). $R[p] \geqslant 0$, $R[G^I] = \alpha > 0$, where α is a finite constant.

Axiom 2 (*Continuity*). R is continuous w.r.t. the weak topology of measures in the following sense: if $\int_{\mathbf{R}^N} p_n \cdot f \to \int_{\mathbf{R}^N} p \cdot f$, $\forall f \in C_0(\mathbf{R}^N)$, then $R[p_n] \to R[p]$.

Axiom 3 (*Translation invariance*). For every $x_0 \in \mathbf{R}^N$, let $p_{x_0}(x) = p(x - x_0)$, $\forall x \in \mathbf{R}^N$, then $R[p_{x_0}] = R[p]$.

Axiom 4 (*Rotation invariance*). For every $N \times N$ orthogonal matrix T, let $p_T(x) = p(T \cdot x)$, $\forall x \in \mathbf{R}^N$, then $R[p_T] = R[p]$.

Axiom 5 (*Luminance invariance*). For every $c \neq 0$, let $p^{[c]} = c \cdot p(x)$, $\forall x \in \mathbf{R}^N$, then $R[p^{[c]}] = R[p]$.

Axiom 6 (*Homogeneous scaling*). For any $\beta > 0$, let $p_{[\beta]}(x) = p(\beta x)$, $\forall x \in \mathbf{R}^N$, then $R[p_{[\beta]}] = \frac{R[p]}{\beta}$.

Axiom 7 (*Combination*). There exists a function F such that for any two imaging systems with PSFs p_1 and p_2, the image resolution of the composite system with the PSF $p = p_1 * p_2$ by serial connection of the two systems is

$$R[p] = R[p_1 * p_2] = F\bigl[R[p_1], R[p_2]\bigr]. \tag{6}$$

The following theorem was proved in [4].

Theorem 1.1 [4]. *If $R[p]$ is an image resolution measure satisfying Axioms* 1–7*, then for any $p \in \mathcal{P}$,*

$$R[p] = \alpha \sqrt{\frac{\mathrm{Trace}(\sigma[p])}{N}}. \tag{7}$$

Although the axioms in [3,4] led to one specific solution to the image resolution quantification problem, one may wonder whether there are other possibilities. From a different perspective, it is important to ask how the image resolution will change under affine transformations, which was missing in [3,4]. However, in multiple dimensions, there is no direct heuristics available for specifying the change of image resolution under affine transformations. Therefore, we propose a quite general axiom to characterize the change of the image resolution under affine transformations:

Axiom A (*Affine transformation*). There exists a function H such that for any $N \times N$ nonsingular matrix T and PSF $p \in \mathcal{P}$, let $p_T = p(T \cdot x)$, $\forall x \in \mathbf{R}^N$, then $R[p_T] = H(R[p], T)$.

Although the Affine Transformation Axiom appears quite general, it is not consistent with Axioms 1–7, cf. Lemma 2.2. It is because the Combination Axiom 7 is rather strong. One remedy to solve this inconsistency is to introduce the following weak form of the Combination axiom:

Axiom W (*Weak combination*). There exists a function J such that for any PSF p, the resolution measure of the imaging system with the PSF $q = p * p$ is

$$R[q] = R[p * p] = J\big(R[p]\big). \tag{8}$$

As it will become clear later, the weaker Axiom W makes the axiomatic system less restrictive. Therefore, general properties of image resolution measures can be proved, and new image resolution measure can be established. More precisely, let $\lambda[p] = (\lambda_1[p], \ldots, \lambda_N[p])$, where $\lambda_j[p]$ is the eigenvalues of $\sigma[p]$, $j = 1, \ldots, N$. The first main result is

Theorem 1.2. *Assume that $R[\cdot]$ is an image resolution measure satisfying Axioms* 1–6 *and* W. *Then, for any $p \in \mathcal{P}$, $R[p]$ is a homogeneous symmetric function of order* $1/2$ *of* $\lambda[p]$:

$$R[p] = g\big(\lambda[p]\big), \tag{9}$$

where $g(\cdot)$ is symmetric and for $t > 0$ and $\lambda \in \mathbf{R}_+^N$,

$$g(t\lambda) = \sqrt{t}\, g(\lambda). \tag{10}$$

Furthermore, when augmented with the Affine Transformation Axiom, the image resolution measure is proved to be proportional to the squared root of the geometric mean of the eigenvalues of the covariance matrix of the PSF, which is the second main result

Theorem 1.3. *Assume that $R[\cdot]$ is an image resolution measure satisfying Axioms* 1–6, W *and* A. *Then, for any $p \in \mathcal{P}$,*

$$R[p] = \alpha \left(\prod_{j=1}^{N} \lambda_j[p] \right)^{\frac{1}{2N}}. \tag{11}$$

The structure of the rest of this paper is as follows. In Section 2, we prove some helpful lemmas. In Sections 3 and 4, we respectively present the proofs of Theorems 1.2 and 1.3 and relevant corollaries. In Section 5, we discuss relevant issues and future directions.

2. Some lemmas

Again, we assume that $R[\cdot]$ is an image resolution measure for imaging systems specified by the admissible PSF set (4). The following lemma follows by direct computation.

Lemma 2.1. *For any $N \times N$ nonsingular matrix T and any PSF $p \in \mathcal{P}$,*

$$\sigma[p_T] = T^{-1}\sigma[p]T^{-1^{\mathrm{tr}}}, \tag{12}$$

where $^{\mathrm{tr}}$ *denotes the transpose of a matrix.*

Lemma 2.2. *Axiom* A *is not consistent with Axioms* 1–7.

Proof. For any $N \times N$ nonsingular matrix T, let $S = T^{-1}$. For any PSF $p \in \mathcal{P}$, we have

$$\operatorname{Trace}\bigl(\sigma[p_T]\bigr) = \sum_{r,s=1}^{N}\left[\sum_{j=1}^{N} S_{j,r}S_{j,s}\right]\sigma[p]_{r,s}. \tag{13}$$

If Axiom A is consistent with Axioms 1–7, then $R[\cdot]$ is given by Theorem 1.1. Then the function H in Axiom A should satisfy the following equation:

$$H\left[\alpha\sqrt{\frac{\sum_{t=1}^{N}\sigma[p]_{t,t}}{N}}, T\right] = \alpha\sqrt{\frac{\sum_{r,s=1}^{N}\bigl[\sum_{j=1}^{N} S_{j,r}S_{j,s}\bigr]\sigma[p]_{r,s}}{N}}. \tag{14}$$

This is impossible because the left-hand side only depends on $\sum_{t=1}^{N}\sigma[p]_{t,t}$ while the right-hand side depends on all components of $\sigma[p]$. □

Lemma 2.3. *If $R[\cdot]$ satisfies Axioms* 1, 5, *and* 6, *then for a Gaussian PSF $G^{\xi\cdot I}$ with covariance matrix $\xi \cdot I$, $\xi > 0$, we have*

$$R\bigl[G^{\xi\cdot I}\bigr] = \alpha \cdot \sqrt{\xi}, \tag{15}$$

where α is the positive constant in Axiom 1.

Proof. Note that $G^{\xi\cdot I}(x) = c \cdot G^{I}\bigl(\frac{1}{\sqrt{\xi}}x\bigr)$, for some positive c. By Axioms 5 and 6, $R[G^{\xi\cdot I}] = \sqrt{\xi} \cdot R[G^{I}]$. The result follows immediately. □

Lemma 2.4. *If $R[\cdot]$ satisfies Axioms* 1, 5, 6, *and* W, *then the function $J(r)$ in Axiom* W *is equal to $\sqrt{2}r$, for $r > 0$.*

Proof. By Axiom 1, $\alpha = R[G^{I}] > 0$. For $r > 0$, let p be the Gaussian PSF defined by $p = G^{(\frac{r}{\alpha})^2\cdot I}$. Then, by Lemma 2.3, $R[p] = \alpha \cdot \frac{r}{\alpha} = r$. Because

$$p * p = G^{((\frac{r}{\alpha})^2+(\frac{r}{\alpha})^2)\cdot I} = G^{(\frac{\sqrt{2}r}{\alpha})^2\cdot I}, \tag{16}$$

we have $R[p * p] = \sqrt{2}r$. Hence, by Axiom W,

$$J(r) = J\big(R[p]\big) = R[p * p] = \sqrt{2}r. \qquad \square \tag{17}$$

Lemma 2.5. *If $R[\cdot]$ satisfies Axiom* W, *there exists a function K such that for any PSF $p \in \mathcal{P}$, the image resolution of the imaging system with the PSF*

$$q = \underbrace{p * p * \cdots * p}_{n \text{ terms}},$$

where $n = 2^k$, $k = 1, \ldots,$ only depends on $R[p]$ and n:

$$R[q] = R[\underbrace{p * p * \cdots * p}_{n \text{ terms}}] = K\big(R[p], n\big). \tag{18}$$

Furthermore, if $R[\cdot]$ satisfies Axioms 1, 5, 6, *and* W, *then for $n = 2^k$, $k = 1, \ldots,$*

$$K(r, n) = \sqrt{n}r, \quad \text{for } r > 0. \tag{19}$$

Proof. Equation (18) is obtained after the recursive use of Axiom W. In fact, it is easy to obtain

$$K(r, n) = \underbrace{J \circ J \circ \cdots \circ J}_{k \text{ terms}}(r).$$

(19) follows easily from Lemma 2.4. $\square$

Lemma 2.6. *Let $R[\cdot]$ satisfy Axioms* 1, 5, 6, *and* W. *For any PSF p, $n = 2^k$, $k = 1, \ldots,$ let*

$$q_n(x) = (\sqrt{n})^N \cdot p(\sqrt{n} \cdot x), \tag{20}$$

$$g_n = \underbrace{q_n * q_n * \cdots * q_n}_{n \text{ terms}}, \tag{21}$$

then

$$R[g_n] = R[p]. \tag{22}$$

Proof. By Axioms 5 and 6, $R[q_n] = \frac{R[p]}{\sqrt{n}}$. By Lemma 2.5, the conclusion follows immediately. $\square$

The following lemma is trivial.

Lemma 2.7. *If $R[\cdot]$ satisfies Axioms* 3 *and* 5, *then $R[p] = R[p_1]$, where $p_1(x) = \frac{1}{\int_{\mathbf{R}^N} p(x)\,dx} \times p(x - \mu)$ and μ is the mean vector defined in* (2).

Thus, *it suffices to consider PSFs that are probability density functions with zero mean vectors, i.e., $\int_{\mathbf{R}^N} p(x)\,dx = 1$ and $\int_{\mathbf{R}^N} x_k p(x)\,dx = 0$.*

3. Proof of Theorem 1.2

Proof. By Lemma 2.7, we assume that p is a probability density function with a zero mean vector. Let g_n be constructed as in Lemma 2.6. By Lemma 2.6, for $n = 2^k$, $k = 1, \ldots,$ $R[g_n] =$

$R[p]$. Let $\hat{p}$ be the Fourier transform of p. Because p is a probabilistic distribution with a zero mean vector, the second order Taylor's expansion of the Fourier transform $\hat{p}$ around $\omega = 0$ is

$$\hat{p}(\omega) = 1 - \frac{1}{2}\omega^{\mathrm{tr}} \cdot \sigma \cdot \omega + o\big(\|\omega\|^2\big). \tag{23}$$

Since $\hat{q}_n(\omega) = \hat{p}\big(\frac{\omega}{\sqrt{n}}\big)$, we have

$$\hat{q}_n(\omega) = 1 - \frac{1}{2n}\omega^{\mathrm{tr}} \cdot \sigma \cdot \omega + o\left(\frac{\|\omega\|^2}{n}\right), \tag{24}$$

for any $\omega \in \mathbf{R}^N$, and n is sufficiently large. Then,

$$\hat{g}_n(\omega) = \hat{q}_n^n = \left(1 - \frac{1}{2n}\omega^{\mathrm{tr}} \cdot \sigma \cdot \omega + o\left(\frac{\|\omega\|^2}{n}\right)\right)^n. \tag{25}$$

Hence,

$$\lim_{n\to\infty} \hat{g}_n(\omega) = \mathbf{e}^{-\frac{1}{2}\omega^{\mathrm{tr}}\cdot\sigma\cdot\omega}, \quad \forall \omega \in \mathbf{R}^N. \tag{26}$$

Since the covariance matrix of p is finite and positive definite, by (26), for any $\omega \in \mathbf{R}^N$,

$$\lim_{n\to\infty} \hat{g}_n(\omega) = \widehat{G^\sigma}(\omega). \tag{27}$$

By [18, Proposition 8.69], g_n converges to G^σ vaguely in the sense of measure as measure density functions. Since all the g_n have the same resolution $R[p]$, by Axiom 2, $R[p] = R[G^{\sigma[p]}]$. Let f be defined by

$$f(\sigma) \triangleq R\big[G^\sigma\big] \tag{28}$$

for any positive definite matrix σ. For any $t > 0$, by Axiom 6,

$$f\big(t \cdot \sigma[p]\big) = R\big[G^{t\cdot\sigma[p]}\big] = R\big[G^{\sigma[p]}_{[\frac{1}{\sqrt{t}}]}\big] = \sqrt{t}R\big[G^{\sigma[p]}\big] = \sqrt{t}\, f\big(\sigma[p]\big). \tag{29}$$

Let T be an orthogonal matrix such that $\sigma[p]$ is diagonalized: $D = T^{\mathrm{tr}}\sigma[p]T$, where D is the diagonal matrix with positive diagonal elements $\lambda_1[p], \dots, \lambda_N[p]$. By Lemma 2.1, the covariance matrix of p_T is equal to $T^{\mathrm{tr}}\sigma[p]T$. By Axiom 4,

$$R[p] = f\big(\sigma[p]\big) = R[p_T] = f(D).$$

The order of $\lambda_1[p], \dots, \lambda_N[p]$ can be freely exchanged by choosing appropriate orthogonal matrix T. Hence, $f(\sigma[p])$ is a symmetric function of $\lambda_1[p], \dots, \lambda_N[p]$. This dependence is expressed by defining a new function $g(\lambda[p]) = f(\sigma[p])$. The property of g can be easily verified. □

The result in [3] can be now established with the weak Combination Axiom W, instead of using the strong Combination Axiom 7.

Corollary 3.1. *Under the same assumptions of Theorem* 1.2, *if* $N = 1$, *then for any* $p \in \mathcal{P}$,

$$R[p] = \alpha\sqrt{\sigma}, \tag{30}$$

where σ *is the variance of* p.

Proof. By Theorem 1.2, $R[p] = f(\sigma)$. Since f is homogeneous, by (29), $f(\sigma) = \sqrt{\sigma} f(1)$. By Axiom 1, $f(1) = \alpha$, which is the same as the finding by Wang and Li [3]. □

It follows as a corollary of Theorem 1.2 that the image resolution measures for some imaging systems can be determined without requiring the strong Combination Axiom 7.

Corollary 3.2. *Under the same assumptions of Theorem* 1.2*, if all eigenvalues of the covariance matrix of p are the same as $\xi > 0$, then for any $p \in \mathcal{P}$,*

$$R[p] = \alpha \cdot \sqrt{\xi}. \tag{31}$$

Proof. This is obtained by Theorem 1.2 and Axiom 1. □

Here is one practical case where the assumptions of Corollary 3.2 hold. If a PSF is radially symmetric, i.e., $p(x) = p(\|x\|)$, $x \in \mathbf{R}^N$, its covariance matrix must be diagonal with the same diagonal elements

$$\xi = \frac{1}{N} \int\limits_{\mathbf{R}^N} \|x\|^2 p\big(\|x\|\big)\, dx. \tag{32}$$

By Corollary 3.2, we have

Corollary 3.3. *Under the same assumptions of Theorem* 1.2*, for a radially symmetric PSF $p \in \mathcal{P}$,*

$$R[p] = \alpha \cdot \sqrt{\xi}. \tag{33}$$

4. Proof of Theorem 1.3

Proof. By the proof of Theorem 1.2, we have $R[p] = R[G^D]$. Let $B = \sqrt{D}$ be the diagonal matrix with diagonal elements $\sqrt{\lambda_1[p]}, \ldots, \sqrt{\lambda_N[p]}$. Then, $G^D = cG^I_B$, where c is a positive constant. Therefore,

$$R[p] = H(\alpha, B).$$

For any PSF $q \in \mathcal{P}$ and any two nonsingular matrices U and V, by Axiom A,

$$H\big(R[q], U \cdot V\big) = R[q_{UV}] = R\big[(q_V)_U\big) = H\big(R[q_V], U\big) = H\big(H\big(R[q], V\big), U\big). \tag{34}$$

Specifically, if U is orthogonal, by Axiom 4,

$$H\big(R[q], U \cdot V\big) = H\big(R[q], V\big). \tag{35}$$

Similarly, if V is orthogonal,

$$H\big(R[q], U \cdot V\big) = H\big(R[q], U\big). \tag{36}$$

Hence, pre- or post-multiplication of an affine transformation T by an orthogonal matrix does not change the value of $H(R[q], T)$. Furthermore, if V is orthogonal, U and W are nonsingular, then

$$H\big(R[q], U \cdot V \cdot W\big) = H\big(H\big(R[q], WV\big), U\big) = H\big(H\big(R[q], W\big), U\big) = H\big(R[q], U \cdot W\big), \tag{37}$$

where the second equality follows by (35). Consequently, if T is a product of nonsingular matrices and orthogonal matrices, those orthogonal matrices can be removed from the product without changing the value of $H[R[q], T]$.

For $1 \leqslant i, j \leqslant N$, let $C(i, j)$ be the matrix obtained by exchanging the ith and jth columns, $R(i, j)$ the matrix by exchanging the ith and jth rows of the identity matrix. For $\xi > 0$, let $S(\xi)$ be the diagonal matrix with the first diagonal element being ξ and other diagonal elements being equal to 1. Then, B can be decomposed into a product of $S(\sqrt{\lambda_1[p]}), \ldots, S(\sqrt{\lambda_N[p]})$, and a finite number of $C(i, j)$'s and $R(i, j)$'s

$$D = S\bigl(\sqrt{\lambda_1[p]}\bigr) \prod_{j=2}^{N} R(1, j) S\bigl(\sqrt{\lambda_j[p]}\bigr) C(1, j). \tag{38}$$

E.g., when $N = 2$, we have

$$\begin{pmatrix} \sqrt{\lambda_1[p]} & \\ & \sqrt{\lambda_2[p]} \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda_1[p]} & \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_2[p]} & \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since each of $C(i, j)'s$ and $R(i, j)'s$ is orthogonal, we have

$$R[p] = H\left(\alpha, \prod_{j=1}^{N} S\bigl(\sqrt{\lambda_j[p]}\bigr)\right) = H\left(\alpha, S\left(\sqrt{\prod_{j=1}^{N} \lambda_j[p]}\right)\right). \tag{39}$$

Therefore, the function $g(\lambda_1[p], \ldots, \lambda_N[p])$ in Theorem 1.2 only depends on the product of $\lambda_1[p], \ldots, \lambda_N[p]$. Let us define a function $\bar{g}$ for this dependency

$$\bar{g}\left(\prod_{j=1}^{N} \lambda_j\right) \triangleq g(\lambda_1, \ldots, \lambda_N), \tag{40}$$

for $\lambda \in \mathbf{R}_+^N$. Since

$$\sqrt{t}\,\bar{g}\left(\prod_{j=1}^{N} \lambda_j\right) = \sqrt{t}\,g(\lambda) = g(t\lambda) = \bar{g}\left(t^N \prod_{j=1}^{N} \lambda_j\right), \tag{41}$$

for $t > 0$, we have

$$\bar{g}\left(t \cdot \prod_{j=1}^{N} \lambda_j\right) = t^{\frac{1}{2N}} \bar{g}\left(\prod_{j=1}^{N} \lambda_j\right). \tag{42}$$

Hence,

$$\bar{g}\left(\prod_{j=1}^{N} \lambda_j\right) = \left(\prod_{j=1}^{N} \lambda_j\right)^{\frac{1}{2N}} \cdot \bar{g}(1). \tag{43}$$

Because $\bar{g}(1) = g(1, \ldots, 1) = \alpha$ by Axiom 1, the conclusion follows immediately. □

5. Discussion

The following comments are in order on the proposed axioms.

(1) At least one PSF must have positive resolution measure to rule out the trivial case $R[p] = 0$ for all p. The constant $\alpha = R[G^I]$ in Axiom 1 serves as the standard unit for the image resolution measure $R[\cdot]$, cf. Theorems 1.1 and 1.3.
(2) The continuity, translation invariance, luminance invariance, scaling, and combination axioms are essentially the same as one-dimensional counterparts in Wang and Li [3]. Luminance invariance is assumed for $c \neq 0$ to allow negative integral $\int_{\mathbf{R}^N} p(x)\,dx$, e.g., in Lemma 2.7. For nonnegative PSF p it can still be assumed to hold only for $c > 0$ as in [3,4].
(3) Rotation invariance assumes all axes are in the same units and no direction is preferred. Directional resolution measures will be studied in the future.
(4) Physically, the combination axiom provides the basis to study the performance of a composite system consisting of serially connected systems. Mathematically, it allows the reduction of a general PSF to a Gaussian distribution with the same first two moments in determining the resolution measure. It should be noted that the function F in Axiom 7 is uniquely determined by the axioms to be $F(x, y) = \sqrt{x^2 + y^2}$ [4], which is consistent with the function J in Axiom W.
(5) It is remarkable that the original finding in [3] can be easily rediscovered with the Combination Axiom W by Theorem 1.2, instead of using the strong combination Axiom 7 in the previous study.
(6) The Affine Transformation Axiom A is consistent with Axioms 1–6 and the weak Combination Axiom W, which together lead to the image resolution measure that is equal to the square root of the geometric mean of the eigenvalues of the covariance matrix of the PSF. On the other hand, Axioms 1–7 confine the image resolution measure to be the square root of the arithmetic mean of the eigenvalues of the covariance matrix of the PSF.

Although the Dirac delta function is not a classical function and hence it is not in the admissible PSF set, we can enlarge the admissible PSF set by considering its closure under the weak topology of measures. The resolution of the Dirac delta function is zero by continuity and Theorem 1.1 or 1.2, or 1.3 because the Dirac delta function is the limit of a sequence of Gaussian functions $G^{\sigma_n \cdot I}$ as σ_n converges to zero in the weak topology of measures.

As seen from the proof of Theorem 1.2, the spatial resolution of such an imaging system is derived by the limiting performance of the composite system consisting of a number of identical systems in serial, which is equivalent to the system behavior near the zero frequency. We acknowledge that this can be a limitation of the present results from a wider perspective. Further studies are needed to understand the implications of the limitation. In other words, it should be interesting to develop an axiomatic resolution theory involving more frequency components. We are actively exploring along this direction.

One improvement to Axiom W was hinted by one anonymous Reviewer. We may consider the following axiom

$$R[p * p] \geqslant \sqrt{2} R[p], \tag{44}$$

to replace (8) because the composite system gives (typically) more blurring. We can establish an inequality for resolution measures by tracing the proofs in the manuscript and replacing all the occurrences of Axiom W by (44). Our specific results are as follows:

- We do not need to derive the form of the function J. Lemma 2.4 is not necessary.
- The result in Lemma 2.5 becomes

$$R[q] \geqslant \sqrt{n} R[p]. \tag{45}$$

- The result in Lemma 2.6 becomes

$$R[g_n] \geqslant R[p]. \tag{46}$$

- In the proof of Theorem 1.2, we have now

$$R\big[G^{\sigma[p]}\big] \geqslant R[p] \tag{47}$$

before the definition of $f(\sigma)$ in (28). The conclusion of Theorem 1.2 is then modified by changing $R[p] = g(\lambda[p])$ to

$$R[p] \leqslant g\big(\lambda[p]\big). \tag{48}$$

- In Corollaries 3.1, 3.2, and 3.3, we just have to replace $=$ by $\leqslant$ in the conclusions.
- In the proof of Theorem 1.3, we need to change $R[p] = R[G^D]$ to $R[p] \leqslant R[G^D]$. Then, we use Gaussian PSFs to replace all the occurrences of general PSFs q or p in the remaining part. The conclusion of Theorem 1.3 is then an inequality after replacing $=$ with $\leqslant$.

It can be concluded that the present results in Theorems 1.1 and 1.3 provide upper bounds for the resolution measures if we use the inequality-based axiom (44). In this setting, it should be possible to specify resolution measures when more axioms are included.

Another important issue suggested by the anonymous reviewer is to develop a resolution theory for the imaging model with noise

$$g = p * f + \varepsilon, \tag{49}$$

where ε is the noise process. In the proof of Theorem 1.2 the noise term will create a difficulty in establishing an appropriate form of the central limit theorem. We emphasize that this difficulty is associated with the mathematical technique we have selected.

Further work is underway to reveal relationships among resolution measures, generalize the theory into the real and complex domains, and apply the findings to help solve real-world problems [16].

References

[1] T.L. Williams, The Optical Transfer Function of Imaging Systems, Institute of Physics Publishing, Bristol and Philadelphia, 1999.
[2] M. Born, E. Wolf, Principles of Optics, seventh ed., Cambridge Univ. Press, 1999.
[3] G. Wang, Y. Li, Axiomatic approach for quantification of image resolution, IEEE Signal Process. Lett. 6 (1999) 257–258.
[4] J.A. O'Sullivan, M. Jiang, X. Ma, G. Wang, Axiomatic quantification of multi-dimensional image resolution, IEEE Signal Process. Lett. 9 (4) (2002) 120–122.
[5] S.M. Blumenfeld, G.H. Glover, Spatial resolution in computed tomography, in: T.H. Newton, D.G. Potts (Eds.), Radiology of the Skull and Brain: Technical Aspects of Computed Tomography, vol. 5, Mosby, 1981.
[6] N. Villain, Y. Goussard, J. Idier, M. Allain, Three-dimensional edge-preserving image enhancement for computed tomography, IEEE Trans. Medical Imaging 22 (2003) 1275–1287.
[7] E.L. Nickoloff, R. Riley, A simplified approach for modulation transfer function determinations in computed tomography, Medical Physics 12 (1985) 437–442.
[8] S. Dore, Y. Goussard, Experimental determination of ct point spread function anisotropy and shift-variance, in: Proc. 19th Annual Int. Conf. of the IEEE, Engineering in Medicine and Biology Society, vol. 2, 1997, pp. 788–791.
[9] S. Rathee, Z.J. Koles, T. Overton, Image restoration in computed tomography: The spatially invariant point spread function, IEEE Trans. Medical Imaging 11 (1992) 530–538.
[10] S. Rathee, Z.J. Koles, T. Overton, Image restoration in computed tomography: Estimation of the spatially variant point spread function, IEEE Trans. Medical Imaging 11 (1992) 539–545.

[11] S. Rathee, Z.J. Koles, T. Overton, Image restoration in computed tomography: restoration of experimental CT images, IEEE Trans. Medical Imaging 11 (1992) 546–553.
[12] K.H. Wong, H.R. Tang, G. Segall, B.H. Hasegawa, Development of quantitative imaging methods for the GE Hawkeye CT/SPECT system, in: Nuclear Science Symposium Conference Record, vol. 4, 2001, pp. 2170–2173.
[13] G. Wang, M.W. Vannier, M.W. Skinner, M.G.P. Cavalcanti, G. Harding, Spiral CT image deblurring for cochlear implantation, IEEE Trans. Medical Imaging 17 (1998) 251–262.
[14] M. Jiang, G. Wang, M.W. Skinner, J.T. Rubinstein, M.W. Vannier, Blind deblurring of spiral CT images, IEEE Trans. Medical Imaging 22 (7) (2003) 837–845.
[15] M. Jiang, G. Wang, M.W. Skinner, J.T. Rubinstein, M.W. Vannier, Blind deblurring of spiral CT images–study of different ratios, Medical Physics 29 (5) (2002) 821–829.
[16] J.F. Meinel, G. Wang, M. Jiang, T. Frei, M.W. Vannier, E.A. Hoffman, Spatial variation of resolution and noise in multi-detector row spiral CT, Academic Radiology 10 (6) (2003) 607–613.
[17] Z.H. Cho, J.P. Jones, M. Singh, Foundations of Medical Imaging, Wiley, New York, 1993.
[18] G.B. Folland, Real Analysis, Modern Techniques and Their Applications, Wiley, New York, 1984.