A Grangeat-type half-scan algorithm for cone-beam CT

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Modern CT and micro-CT scanners are rapidly moving from fan-beam toward cone-beam geometry. Half-scan CT algorithms are advantageous in terms of temporal resolution, and widely used in fan-beam and cone-beam geometry. While existing half-scan algorithms for cone-beam CT are in the Feldkamp framework, in this paper we compensate missing data explicitly in the Grangeat framework, and formulate a half-scan algorithm in the circular scanning case. The half-scan spans 180° plus two cone angles that guarantee sufficient data for reconstruction of the midplane defined by the source trajectory. The smooth half-scan weighting functions are designed for the suppression of data inconsistency. Numerical simulation results are reported for verification of our formulas and programs. This Grangeat-type half-scan algorithm produces excellent image quality, without off-mid-plane artifacts associated with Feldkamp-type half-scan algorithms. The Grangeat-type half-scan algorithm seems promising for quantitative and dynamic biomedical applications of CT and micro-CT. © 2003 American Association of Physicists in Medicine. [DOI: 10.1118/1.1562941]

Key words: Computed tomography (CT), cone-beam geometry, half-scan, Grangeat-type reconstruction

I. INTRODUCTION

Modern CT and micro-CT scanners are rapidly moving from fan-beam toward cone-beam geometry. The half-scan mode is valuable, because it shortens the data acquisition time and improves temporal resolution. Hence, half-scan CT algorithms are widely used in fan-beam and cone-beam geometry. While existing half-scan algorithms for cone-beam CT are in the Feldkamp framework, we propose a Grangeat-type half-scan algorithm in the circular scanning case. Our motivation is to perform appropriate data filling using the Grangeat approach, and suppress the off-mid-plane artifacts associated with the Feldkamp-type algorithms. Interestingly, it came to our attention that Noo and Heuscher just published a half-scan cone-beam reconstruction paper in a SPIE conference. Their work is also based on the Grangeat algorithm but it is in the filtered back-projection framework, while ours is in the rebinning framework. The differences between the two Grangeat-type half-scan algorithms will be highlighted in Sec. IV.

This paper is organized as follows. In the next section, we modify Grangeat’s formula for circular half-scan geometry, by combining redundant data to improve contrast resolution while optimizing temporal resolution. The redundant data are weighted with smooth functions to suppress data inconsistency. In Sec. III, we describe our software simulator and experimental design, and include images reconstructed from simulated projections of a spherical phantom and the Shepp–Logan phantom. In the last section, we discuss relevant issues, and conclude the paper.

II. MATERIALS AND METHODS

A. Grangeat framework

As shown in Fig. 1, the radon transform of a three-dimensional (3-D) function $f(x)$ is defined by

$$Rf(\rho n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tilde{x}) \delta(\tilde{x} \cdot n - \rho) d\tilde{x},$$  \hspace{1cm} (1)

where $n$ is the unit vector that passes through the characteristic point $C$ described by spherical coordinates $(\rho, \theta, \phi)$, $\tilde{x}$ Cartesian coordinates $(x, y, z)$. Equation (1) means that the radon value at $C$ is the integral of the object function $f(\tilde{x})$ on the plane through $C$ and normal to the vector $\tilde{n}$. It is well known that the 3-D function $f(\tilde{x})$ can be reconstructed from $Rf(\rho n)$ provided that $Rf(\rho \tilde{n})$ is available for all planes through a neighborhood of point $\tilde{x}$. The inversion formula of the 3-D radon transform is given by

$$f(\tilde{x}) = -\frac{1}{8\pi^2} \int_{\theta=0}^{\pi/2} \int_{\phi=-\pi/2}^{\pi/2} \int_{\theta=0}^{\pi/2} \frac{\partial^2}{\partial \rho^2} Rf[(\tilde{x} \cdot n) \tilde{n}] |\sin \theta| d\phi d\theta.$$  \hspace{1cm} (2)

For cone-beam CT, it is instrumental to connect cone-beam data to 3-D radon data. Smith, Tuy, and Grangeat independently established such connections. Grangeat’s formulation is geometrically attractive and becomes popular. Mathematically, as shown in Fig. 2, the link can be expressed as follows:
\[
\frac{\partial}{\partial \rho} Rf(\rho \vec{n}) = R'f(\rho \vec{n}) \\
= \frac{1}{\cos^2 \beta} \frac{\partial}{\partial \alpha} \int_{-\infty}^{\infty} \frac{SO}{SA} Xf[s(\rho \vec{n}), t, \psi(\rho \vec{n})] dt \\
= \frac{1}{\cos^2 \beta} \frac{\partial}{\partial s} \int_{-\infty}^{\infty} Xw[f[s(\rho \vec{n}), t, \psi(\rho \vec{n})] dt, \quad (3)
\]

where \(Xf[s(\rho \vec{n}), t, \psi(\rho \vec{n})]\) is the detector value the distance \(s\) away from the detector center \(O\) along the line \(t\) perpendicular to \(OCD\) on the detector plane \(D_\phi\) located at the angle \(\psi\) from the \(y\) axis, \(SO\) denotes the distance between the source and the origin, \(SA\) the distance between the source and an arbitrary point \(A\) along \(t\), \(\beta\) the angle between the line \(SO\) and \(SC\), and \(Xw[f[s(\rho \vec{n}), t, \psi(\rho \vec{n})] = (SO/SA)Xf[s(\rho \vec{n}), t, \psi(\rho \vec{n})]\). Given a characteristic point \(C\) in the radon domain, the plane orthogonal to the vector \(\rho \vec{n}\) is determined. Then, the intersection point(s) of the plane with the source trajectory \(S(\lambda)\) can be found, and the detector plane(s) \(D_\phi\) specified, on which the line integration can be performed. Let \(C_D\) denote the intersection of the detector plane \(D_\phi\) with the ray that comes from \(S(\lambda)\) and goes through \(C\). The position \(C_D\) can be described by a vector \(s\vec{n}_D\). To compute the derivative of the radon value at \(C\), the line integration is performed along \(t\), which is orthogonal to the vector \(s\vec{n}_D\).

For a digital implementation of Eq. (3), the derivative in the \(s\) direction is reformulated as the sum of its horizontal and vertical components,

\[
\frac{\partial}{\partial s} Xw[f[s(\rho \vec{n}), t, \psi(\rho \vec{n})] \\
= \cos[\alpha(\rho \vec{n})] \frac{\partial}{\partial p} Xw[f[s(\rho \vec{n}), t, \psi(\rho \vec{n})] \\
+ \sin[\alpha(\rho \vec{n})] \frac{\partial}{\partial q} Xw[f[s(\rho \vec{n}), t, \psi(\rho \vec{n})], \quad (4)
\]

where \(p\) and \(q\) are the Cartesian axes, and \(s\) and \(\alpha\) define a polar system on the detector plane [see Fig. 2(c)]. Substituting Eq. (4) into Eq. (3), we have

\[
R'f(\rho \vec{n}) = \frac{1}{\cos^2 \beta} \left[ \cos[\alpha(\rho \vec{n})] \int_{-\infty}^{\infty} \frac{\partial}{\partial p} Xw[f[s(\rho \vec{n}), t, \psi(\rho \vec{n})] dt \\
+ \sin[\alpha(\rho \vec{n})] \int_{-\infty}^{\infty} \frac{\partial}{\partial q} Xw[f[s(\rho \vec{n}), t, \psi(\rho \vec{n})] dt \right], \quad (5)
\]
where \( \cos \beta = \text{SO/SC}_D \). The detailed geometrical relationship between \((\rho, \theta, \varphi)\) and \((s, \alpha, \psi)\) can be found in Refs. 7 and 13.

### B. Grangeat-type half-scan formula

We modify Eq. (3) into the following half-scan version. A more detailed explanation can be found in Appendix A:

\[
\frac{\partial}{\partial \rho} Rf(\rho \vec{n}) = \frac{2}{i=1} \omega_i(\rho \vec{n}) \frac{1}{\cos \beta} \frac{\partial}{\partial s} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{SO} \, Xf(s(\rho \vec{n}), t, \psi_i(\rho \vec{n})) dt \right],
\]

where

\[
\psi_1(\rho, \theta, \varphi) = \varphi + \sin^{-1} \left( \frac{\rho}{\text{SO} \, \sin \theta} \right),
\]

\[
\psi_2(\rho, \theta, \varphi) = \varphi + \pi - \sin^{-1} \left( \frac{\rho}{\text{SO} \, \sin \theta} \right).
\]

\( \omega_1(\rho, \theta, \varphi) \) and \( \omega_2(\rho, \theta, \varphi) \) are smooth weighting functions to be explained in Sec. II.F. The half-scan geometry is depicted in Fig. 3, which is the central plane where the source trajectory resides. The cone angle is denoted as \( \gamma_m \), defined as half the full cone angle. The scanning angle \( \psi \) varies from 0 to \( \pi + 2 \gamma_m \). The horizontal axis of a detector plane is denoted as \( \rho \).

In the circular full-scan case, for any characteristic point not in the shadow zone or on its surface there exist a pair of detector planes specified by Eqs. (7) and (8). However, in the circular half-scan case, such the dual planes are not always available. When the dual planes are found, we are in “a doubly sampled zone,” when one of them is missing due to the half-scan, we are in “a singly sampled zone.” Of course, we do not have any associated detector plane in the shadow zone. Relative to the full scan, the shadow zone is increased due to the half-scan. The shadow zone area on a meridian plane depends on the angle \( \varphi \) of the meridian plane.
C. Boundaries between singly and doubly sampled zones

The boundary equations between these zones are instrumental for the design of weighing functions \( \omega_1(\rho, \theta, \varphi) \) and \( \omega_2(\rho, \theta, \varphi) \), as well as interpolation and extrapolation into the shadow zone. The term \( \sin^{-1}[\rho/(SO \sin \theta)] \) in Eqs. (7) and (8) is the angular difference between the meridian plane \( M_\varphi \) and the detector plane \( D_\varphi \), where the line integration is performed. This angular difference ranges from \(-90^\circ\) to \(+90^\circ\). Given a half-scan range and a meridian angle \( \varphi \), it can be determined whether \( D_\varphi \) and \( D_{\varphi_2} \) are available by evaluating \( \sin^{-1}[\rho/(SO \sin \theta)] \). It is emphasized that the following geometric property is important to understand the relationship between singly and doubly sampled zones. For a given characteristic point, the normal directions of the two associated detector planes must be symmetric to the normal direction of the meridian plane containing that characteristic point. Let us denote that \( A(\varphi) \) as the critical angle which separates the singly and doubly sampled zones, as shown in Fig. 4. Then, the number of the available detector planes changes from two to one or one to two across this critical angle. Specifically, for the meridian plane of an angle \( \varphi \), the boundary between the singly and doubly sampled zones is expressed as \( \sin^{-1}(\rho/(SO \sin \theta)) = A(\varphi) \). Therefore, we have

\[
\rho_b(\theta, \varphi) = SO \sin A(\varphi) \sin \theta. \tag{9}
\]

The critical angle function \( A(\varphi) \) takes different forms depending on the meridian angle \( \varphi \). We have the following four intervals:

\begin{align*}
I_1: & \quad 0 \leq \varphi < \gamma_m, \\
I_2: & \quad \gamma_m \leq \varphi < 90, \\
I_3: & \quad 90 \leq \varphi < 90 + 2 \gamma_m, \\
I_4: & \quad 90 + 2 \gamma_m \leq \varphi < 180.
\end{align*}

If a pair of detector planes, \( D_{\varphi, A(\varphi)} \) and \( D_{\varphi + A(\varphi)} \), are available, there exist two redundant data for \((\rho, \theta, \varphi)\). If only either of them is available, there exists only one data. If none of them is available, there exists no data. In Fig. 4, the availability of \( D_{\varphi, A(\varphi)} \) and \( D_{\varphi + A(\varphi)} \) are marked, showing whether both are available, only one of them is available, or none of them are available for a given meridian plane \( M_\varphi \). There are the four intervals in Fig. 5. For \( I_1 \), both \( D_{\varphi, A(\varphi)} \) and \( D_{\varphi + A(\varphi)} \) are available when \( \varphi = A(\varphi) = \pi/2 \). Only \( D_{\varphi + A(\varphi)} \) is available when \( 2 \gamma_m - \varphi = A(\varphi) < -\varphi \). Therefore, the boundary between double and single region is formed at \( A_{11}(\varphi) = -\varphi \). Hence, the boundary equation becomes \( \rho_s(\theta, \varphi) = SO \sin(-\varphi) \sin \theta \) from Eq. (9). In the same manner, the critical angle functions can be acquired for the other intervals and are expressed as follows:

\begin{align*}
I_1: & \quad A_{11}(\varphi) = -\varphi, \\
I_2: & \quad A_{12}(\varphi) = \varphi - 2 \gamma_m, \\
I_3: & \quad A_{13}(\varphi) = \varphi - 2 \gamma_m, \quad \text{and} \quad A_{13}(\varphi) = \varphi - \pi, \\
I_4: & \quad A_{14}(\varphi) = \varphi - \pi.
\end{align*}

These critical angle functions are plotted in Fig. 5. Please note that there exist two critical functions for \( I_3 \), therefore, there exist two boundaries in this interval, given as \( \rho_b(\theta, \varphi) = SO \sin(A_{13}(\varphi)) \sin \theta \) and \( \rho_b(\theta, \varphi) = SO \times \sin(A_{13}(\varphi)) \sin \theta \), respectively.

D. Shadow zone boundaries

The shadow zone boundaries are needed to interpolate/extrapolate for missing data. It is known that the boundary equation for the shadow zone in the full-scan case is given by

\[
\rho_s = SO \sin \theta. \tag{11}
\]

In other words, each detector plane gives a radon circle of diameter \( SO \) on the meridian plane. However, in the half-scan case, the diameter of the radon circle may change, depending on the meridian plane angle, as shown in Fig. 6. Specifically, given a meridian plane, we can find both the diameters of the two radon circles on the plane. Then, we can express the shadow zone boundaries as follows:

\begin{align*}
\rho_s &= SO \sin \theta, \quad \rho_s < 0, \\
\rho_s &= SO \sin \theta, \quad \rho_s > 0.
\end{align*}

In reference to Fig. 6, it is easy to verify that

\[
x_0 = SO/2, \quad y_0 = 0, \tag{12}
\]

\[
x_1 = (SO/2) \cos(2 \gamma_m + \pi), \quad y_1 = (SO/2) \sin(2 \gamma_m + \pi).
\]

Then, it can be geometrically derived in each of the four intervals that

\begin{align*}
I_1: & \quad SO_1(\varphi) = \frac{2x_1 - 2y_1 \cot \varphi}{|cosec \varphi|}, \quad SO_1(\varphi) = SO; \\
I_2: & \quad SO_1(\varphi) = \frac{2x_2 - 2y_0 \cot \varphi}{|cosec \varphi|}, \quad SO_1(\varphi) = SO;
\end{align*}
I3: $SO_l(\varphi) = SO, \; SO_r(\varphi) = SO$;
I4: $SO_l(\varphi) = SO, \; SO_r(\varphi) = \frac{2x_1 - 2y_1 \cot \varphi}{|\csc \varphi|}$.  \hfill (14)

E. Weighting functions

Using the above boundary equations, we can graphically depict the singly and doubly sampled zones, along with the shadow zones. Four maps from different meridian planes are represented in Fig. 7. White area stands for the doubly sampled region, gray for the singly sampled region, and black for the shadow zone. When data are consistent, we can set $\omega_1 = \omega_2 = \frac{\pi}{2}$ for the doubly sampled region, set the valid one to 1 and the other as 0 for the singly sampled region, and set both to zero for the shadow zone, where missing data will be estimated afterward. When data are inconsistent in practice, the following smooth weighting functions are designed according to Parker’s half-scan weighting scheme:\(^3\)

$$\omega_1(\rho, \theta, \varphi) = \cos^2 \left( \frac{\pi}{2} \frac{\rho - (\delta) - \rho_b(\theta)}{\rho_b(\theta) - (-\delta)} \right),$$

$$\theta < 0, \quad \rho < \rho_b(\theta);$$

$$0, \quad \theta \geq 0, \quad \rho \geq \rho_b(\theta);$$

$$\sin^2 \left( \frac{\pi}{2} \frac{\rho - \rho_b(\theta)}{\delta - \rho_b(\theta)} \right), \quad \theta \geq 0, \quad \rho \geq \rho_b(\theta).$$

$$\omega_2(\rho, \theta, \varphi) = \sin^2 \left( \frac{\pi}{2} \frac{\rho - (\delta) - \rho_b(\theta)}{\rho_b(\theta) - (-\delta)} \right),$$

$$\theta < 0, \quad \rho < \rho_b(\theta);$$

$$1, \quad \theta < 0, \quad \rho \geq \rho_b(\theta);$$

$$\cos^2 \left( \frac{\pi}{2} \frac{\rho - \rho_b(\theta)}{\delta - \rho_b(\theta)} \right), \quad \theta \geq 0, \quad \rho \geq \rho_b(\theta).$$

Fig. 6. Shadow zone geometry. (a) The midplane view; (b) meridian plane view.

Fig. 7. Maps of singly and doubly sampled zones, as well as shadow zones on different meridian planes. The white area stands for doubly sampled regions, gray for singly sampled regions, and black for shadow zones (a) 7°; (b) 40° (c) 103° (d) 170°.
\[ \omega_2(r, \theta, \varphi) = \frac{\cos^2 \left( \frac{\pi}{2} \frac{\rho - (-\delta)}{\rho_b(\theta) - (-\delta)} \right)}{D}, \]

\[ \theta < 0, \quad \rho < \rho_b(\theta); \]

0, \quad \theta < 0, \quad \rho > \rho_b(\theta); \]

0, \quad \theta > 0, \quad \rho < \rho_b(\theta); \]

\[ \sin^2 \left( \frac{\pi}{2} \frac{\rho - \rho_b(\theta)}{\rho_b(\theta) - (-\delta)} \right), \quad \theta > 0, \quad \rho > \rho_b(\theta). \]

I3: \[ \omega_1(r, \theta, \varphi) = \sin^2 \left( \frac{\pi}{2} \frac{\rho - \rho_b(\theta)}{\rho_b(\theta) - (-\delta)} \right), \]

\[ \theta < 0, \quad \rho < \rho_b(\theta); \]

1, \quad \theta < 0, \quad \rho > \rho_b(\theta); \]

0, \quad \theta > 0, \quad \rho > \rho_b(\theta); \]

\[ \cos^2 \left( \frac{\pi}{2} \frac{\rho - \rho_b(\theta)}{\rho_b(\theta) - (-\delta)} \right), \quad \theta > 0, \quad \rho > \rho_b(\theta). \]

1, \quad \theta > 0, \quad \rho < \rho_b(\theta); \]

\[ \sin^2 \left( \frac{\pi}{2} \frac{\rho - \rho_b(\theta)}{\rho_b(\theta) - (-\delta)} \right), \quad \theta < 0, \quad \rho > \rho_b(\theta). \]

FIG. 8. Smooth weighting functions for the four maps shown in Fig. 7. The value ranges from 0 (black) to 1 (white) (a) \( \omega_1 \), (b) \( \omega_2 \).
In the above weighting functions, the weighting functions are shown in Fig. 8.

TABLE I. Parameters of the phantoms used in our numerical simulation.

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<th>Phantom</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>θ</th>
<th>ϕ</th>
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F. Interpolation/extrapolation

In this study, we use the zero-padding and linear interpolation methods, respectively, to estimate missing data in the shadow zone. Grangeat reconstruction after zero padding is theoretically equivalent to the Feldkamp-type reconstruction.14 The heuristics behind our choice of the linear interpolation method is that the derivative radon of any ellipsoid is linear.15 A few of the radon values near the shadow zone boundary can be linearly interpolated along the θ direction. The zero padding and linear interpolation is shown in Fig. 9.

Of course, there are multiple possibilities for interpolation/extrapolation into the shadow zone. Some other techniques are reported in our previous paper.16 Knowledge-based interpolation/extrapolation is also feasible in this half-scan Grangeat framework. In addition, the parallel-beam approximation of cone-beam projection data suggested by Noo and Heuscher can be applied in this rebinning framework as well.9 In this study, we believe that the linear interpolation technique is sufficient to show the merit of the algorithm.

G. Description of the algorithm

To summarize, the Grangeat-type half-scan algorithm can be implemented in the following steps.

(1) Specify a characteristic point (ρ, θ, ϕ), where the derivative of radon data can be calculated.
(2) Determine the line integration points for the given characteristic point according to Eqs. (7), (8), and the rebinning equations.7,13
(3) Calculate the derivatives of radon data with Eqs. (4) and (5).
(4) Calculate the smooth weighting functions ω1(ρ, θ, ϕ) and ω2(ρ, θ, ϕ).
(5) Apply the weighting functions to the radon data using Eq. (6).
(6) Repeat Steps (1)–(5) until we are done with all the characteristic points that can be calculated.
(7) Estimate the radon data in the shadow zone.

![Image](60x582 to 289x742)
(8) Use the two stage parallel-beam backprojection algorithm according to Eq. (2).\textsuperscript{17,18}

Note that the computational time of the Grangeat-type half-scan algorithm is quite comparable to that of the full-scan Grangeat algorithm. The half-scan algorithm only requires about half of the full-scan range but it uses redundant data. The computational overhead for multiplication of the weighting functions is negligible compared to the total computational time.

III. RESULTS

We developed a software simulator in the IDL Language (Research Systems Inc., Boulder, Colorado) for Grangeat-type image reconstruction. In the implementation of the Grangeat formula, the numerical differentiation is performed with a built-in function based on three-point Lagrangian interpolation. The source-to-origin distance was set to 3.92. The number of detectors per cone-beam projection was 256 by 256. The size of the 2-D detector plane was 2.1 by 2.1. A half of the full-cone angle is about 15°. The number of projections was 360. The number of meridian planes was 180. The numbers of radial and angular samples was 256 by 360, respectively. Each reconstructed image volume had dimensions of 2.1 by 2.1 by 2.1, and contained 256 by 256 by 256 voxels. The numbers of samples were chosen to be greater than the lower bounds we established using the Fourier analysis method.\textsuperscript{16} Both the spherical phantom and the 3-D Shepp–Logan phantom as shown in Table I were used in the numerical simulation.

Figure 10 shows the results obtained from the $y = 0$ plane of the sphere phantom. The half-scan algorithm cannot produce an exact reconstruction for a general object due to the incompleteness of projection data but in special cases like a sphere phantom this algorithm can achieve exact reconstruction with linear interpolation. When data in the shadow zone are linearly interpolated from the half-scan data of a sphere, an exact reconstruction of the sphere can be achieved because the radon transform of a perfect sphere is linear.\textsuperscript{15} This exactness seems impossible with other half-scan cone-beam reconstruction algorithms. Nevertheless, this property is de-
We believe that both half-scan algorithms are complementary. It seems that data filling mechanism is more flexible in our radon-rebinning framework. Noo and Heuscher suggested that the parallel-beam approximation of cone-beam projection data be used to estimate missing data, which is done in the spatial domain. This kind of spatial domain processing is also allowed in our framework. In addition to the spatial domain approximation, the radon domain estimation, such as linear interpolation, spline interpolation and knowledge-based interpolation, can be done in our framework as well. However, in the filtered backprojection framework, each frame of cone-beam projection data can be processed as soon as it is acquired, a desirable property for practical implementation. Clearly, a systematic comparison of the two algorithms is worthy of further investigation.

In conclusion, we have formulated a Grangeat-type half-scan algorithm in the circular scanning case. The half-scan spans 180° plus two cone angles, allowing exact in-plane reconstruction. The smooth half-scan weighting functions have been designed for the suppression of data inconsistency. Numerical simulation results have verified the correctness and demonstrated the merits of our formulas. This Grangeat-type half-scan algorithm is considered promising for quantitative and dynamic biomedical applications of CT and micro-CT. The extension of this work to the helical geometry is being actively pursued. We will also study other promising reconstruction methods in the future.

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APPENDIX: DERIVATION OF THE GRANGEAT-TYPE HALF-SCAN ALGORITHM
Grangeat found a link between radon data and cone-beam projection data, which is expressed as follows:

\[
\frac{\partial}{\partial \rho} Rf(\rho \tilde{n}) = R' f(\rho \tilde{n})
\]

\[
= \frac{1}{\cos^2 \beta} \frac{\partial}{\partial s} \int_{S_0}^{\infty} Xf(t, \psi(\rho \tilde{n})) dt
\]

\[
= \frac{1}{\cos^2 \beta} \frac{\partial}{\partial s} \int_{-\infty}^{\infty} Xw(t, \psi(\rho \tilde{n})) dt.
\]

In the circular full-scan case, there always exist two detector planes from which the radon data can be calculated, except for the shadow zone. The detector planes are specified by the following two equations:

\[
\psi_{\ell}(\rho, \theta, \varphi) = \varphi + \sin^{-1} \left( \frac{\rho}{SO \sin \theta} \right),
\]

\[
\psi_{s}(\rho, \theta, \varphi) = \varphi + \sin^{-1} \left( \frac{\rho}{SO \sin \theta} \right).
\]
\[ \psi_2(\rho, \theta, \varphi) = \varphi + \pi - \sin^{-1}\left(\frac{\rho}{SO \sin \theta}\right). \quad (A3) \]

In an ideal situation where data is noise-free and the object is stationary during the rotation, we have

\[
\frac{1}{\cos^2 \beta} \frac{\partial}{\partial s} \int_{-\infty}^{\infty} \frac{SO}{SA} X_N f(s(\rho \bar{n}), t, \psi_1(\rho \bar{n})) dt
= \frac{1}{\cos^2 \beta} \frac{\partial}{\partial s} \int_{-\infty}^{\infty} \frac{SO}{SA} X_N f(s(\rho \bar{n}), t, \psi_2(\rho \bar{n})) dt. \quad (A4)
\]

Therefore, the radon data can be calculated from either of the detector planes. However, in practice, we should consider the following practical conditions: (1) Projection data contain noise; (2) data from \( D_{\phi_1} \) and \( D_{\phi_2} \) may be different due to motion of an object.

Hence, we have

\[
\frac{1}{\cos^2 \beta} \frac{\partial}{\partial s} \int_{-\infty}^{\infty} \frac{SO}{SA} X f(s(\rho \bar{n}), t, \psi_1(\rho \bar{n})) dt
\neq \frac{1}{\cos^2 \beta} \frac{\partial}{\partial s} \int_{-\infty}^{\infty} \frac{SO}{SA} X f(s(\rho \bar{n}), t, \psi_2(\rho \bar{n})) dt. \quad (A5)
\]

Considering noise in projection data, we should combine redundant data to improve the signal-to-noise ratio. Consequently, Eq. (A5) can be modified to the following:

\[
\frac{\partial}{\partial \rho} Rf(\rho \bar{n}) = \sum_{i=1}^{2} \omega_i(\rho \bar{n}) \frac{1}{\cos^2 \beta} \frac{\partial}{\partial s} \int_{-\infty}^{\infty} \frac{SO}{SA} X f(s(\rho \bar{n}), t, \psi_i(\rho \bar{n})) dt, \quad (A6)
\]

where \( \omega_1(\rho \bar{n}) + \omega_2(\rho \bar{n}) = 1 \). To maximize the signal to noise ratio, both \( \omega_1(\rho \bar{n}) \) and \( \omega_2(\rho \bar{n}) \) should equal to 1/2.\(^\dagger\) However, when motion is significant, one of the solutions to sup-

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**Fig. 12.** Grangeat-type reconstruction of the 3-D Shepp–Logan phantom. (a) Full-scan reconstruction; (b) half-scan reconstruction; (c) comparative profiles from the identical positions marked by the dashed line in (a). (Contrast range: 1.005–1.05.)
press the motion artifacts is to use a half-scan. But in this case, data are not always redundant by a factor of 2. More exactly, there are generally doubly, singly sampled regions, and shadow zones on a meridian plane. Therefore, there exist discontinuities between the adjacent regions. Hence, the smooth weighting functions such as those in Sec. II E are needed to suppress the associated artifacts.

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