PLINE-BASED IMAGE RECONSTRUCTION IN HELICAL CONE-BEAM COMPUTED TOMOGRAPHY WITH A VARIABLE PITCH

Yu Zou, Xiaochuan Pan, and Dan Xia
Department of Radiology, The University of Chicago, Chicago, Illinois 60637
Ge Wang
Department of Radiology, The University of Chicago, Chicago, Illinois 60637
and Department of Radiology, University of Iowa, Iowa City, Iowa 52242

(Received 9 November 2004; revised 12 March 2005; accepted for publication 16 March 2005; published 29 July 2005)

Current applications of helical cone-beam computed tomography (CT) involve primarily a constant pitch where the translating speed of the table and the rotation speed of the source-detector remain constant. However, situations do exist where it may be more desirable to use a helical scan with a variable translating speed of the table, leading a variable pitch. One of such applications could arise in helical cone-beam CT fluoroscopy for the determination of vascular structures through real-time imaging of contrast bolus arrival. Most of the existing reconstruction algorithms have been developed only for helical cone-beam CT with constant pitch, including the backprojection-filtration (BPF) and filtered-backprojection (FBP) algorithms that we proposed previously. It is possible to generalize some of these algorithms to reconstruct images exactly for helical cone-beam CT with a variable pitch. In this work, we generalize our BPF and FBP algorithms to reconstruct images directly from data acquired in helical cone-beam CT with a variable pitch. We have also performed a preliminary numerical study to demonstrate and verify the generalization of the two algorithms. The results of the study confirm that our generalized BPF and FBP algorithms can yield exact reconstruction in helical cone-beam CT with a variable pitch. It should be pointed out that our generalized BPF algorithm is the only algorithm that is capable of reconstructing exactly region-of-interest image from data containing transverse truncations. © 2005 American Association of Physicists in Medicine. [DOI: 10.1118/1.1902530]

I. INTRODUCTION

For helical cone-beam CT with a constant pitch, algorithms based upon Grangeat’s formula have been developed for image reconstruction from the image’s three-dimensional (3D) Radon transform converted from the helical cone-beam projections. Alternative algorithms have also been proposed that reconstruct images directly from helical cone-beam projections, and they have been extended to address approximately image reconstruction from 3-Pi helical cone-beam projections. The PI-line concept plays an important role in image reconstruction from helical cone-beam projections. Recently, we have developed two algorithms for image reconstruction from helical cone-beam projections. One of the two algorithms is referred to as a backprojection-filtration (BPF) algorithm because it reconstructs an image by first backprojection of data derivatives and then filtration of the backprojections on PI-lines, whereas the other is referred to as a filtered-backprojection (FBP) algorithm because it reconstructs an image by first filtering the modified data along the cone-beam projections of the PI-lines onto the detector plane and then backprojecting the filtered data onto PI-line segments.

Many applications of helical cone-beam computed tomography (CT) involve only a constant pitch. However, situations do arise where it may be more desirable to use a helical cone-beam scan with a variable pitch for achieving optimal imaging results. One of such applications could be in helical CT fluoroscopy for the determination of vascular structures through real-time imaging of contrast bolus arrival. The use of a variable pitch offers the opportunity to synchronize the scanning aperture with the moving bolus peak for acquisition of data with maximized information. Therefore, it is of practical merit to develop algorithms for image reconstruction in this situation.

One of the existing algorithms developed for helical scan with a constant pitch has recently been generalized to reconstruct images exactly in helical cone-beam CT with a variable pitch. As will be seen below, the perspective of PI-lines also provides a natural basis for investigating the problem of image reconstruction in helical cone-beam scan. In this work, we generalize our PI-line-based algorithms to reconstruct images in helical cone-beam CT with a variable pitch. This paper is organized as follows. In Sec. II, our PI-line-based BPF and FBP algorithms are generalized. In Sec. III, we present the numerical results of our preliminary study to evaluate and validate the generalized BPF and FBP algorithms. In Sec. IV, we make remarks on the implications and generalization of this work.

II. THEORY

In helical cone-beam CT with a constant pitch, we have developed a general formula for image reconstruction on PI-line segments. Based upon this formula, we subsequently derived two reconstruction algorithms, which are referred to...
as the BPF and FBP algorithms, respectively. In this section, we show that, under the condition on the variable pitch, our general formula remains valid. Once its validity is established, we use it to derive the FBP and BPF algorithms for image reconstruction from data acquired by use of the helical cone-beam scan with the variable pitch.

We first introduce the fixed-coordinate system \{x, y, z\} and the rotation-coordinate system \{u_0, v_0, w_0\}, which are fixed on the center of rotation and on the rotating source point, respectively. For a rotation angle \(\lambda\), we use \(\hat{e}_x(\lambda)\), \(\hat{e}_y(\lambda)\), and \(\hat{e}_z(\lambda)\) to depict the unit vectors of the rotation-coordinate system, which can be expressed as \(\hat{e}_x(\lambda)=(-\sin \lambda, \cos \lambda, 0)^T\), \(\hat{e}_y(\lambda)=(0, 1, 0)^T\), and \(\hat{e}_z(\lambda)=(\cos \lambda, \sin \lambda, 0)^T\) in the fixed-coordinate system.

In a helical scan with a constant radius \(R\), the fixed and rotation coordinates are related through

\[
x = -u_0 \sin \lambda + (w_0 + R) \cos \lambda,
\]
\[
y = u_0 \cos \lambda + (w_0 + R) \sin \lambda,
\]
\[
z = v_0 + Z(\lambda),
\]
where \(Z(\lambda)\) is a function of \(\lambda\), and its first order derivative \(Z'(\lambda)=dZ(\lambda)/d\lambda\) is referred to as the variable pitch. Also, we consider a detector plane that is parallel to the \(u_0-v_0\) plane and is at a distance \(S\) from the source point. Let \(u\) and \(v\) denote the cone-beam projections of \(u_0\) and \(v_0\) onto the detector plane, then we have

\[
u_0 = -\frac{w_0}{S}u \quad \text{and} \quad v_0 = -\frac{w_0}{S}v.
\]

A. The general cone-beam reconstruction formula

In the fixed-coordinate system, the source trajectory in a helical scan can be expressed as

\[
r_0(\lambda) = (R \cos \lambda, R \sin \lambda, Z(\lambda))^T.
\]

Obviously, when \(Z'(\lambda)=h/2\pi\) is chosen, where \(h\) is a constant, one obtains the conventional helical scan with a constant pitch \(h/2\pi\).

Let \(f(r)\) denote the 3D object function that has a compact support enclosed entirely within the helical trajectory. The cone-beam projection of \(f(r)\) onto the detector plane can be expressed mathematically as

\[
D(r_0(\lambda), \hat{\beta}(r', \lambda)) = \int_0^\infty dt f(r_0(\lambda) + t\hat{\beta}(r', \lambda)),
\]
where \(r' \in \mathbb{R}^3\), and the unit vector \(\hat{\beta}\) denotes the direction of a particular ray passing through \(r'\) and is given by

\[
\hat{\beta}(r', \lambda) = \frac{r' - r_0(\lambda)}{|r' - r_0(\lambda)|}.
\]

Because the image support is entirely enclosed by the helix, we have \(D(r_0(\lambda), \hat{\beta}(r', \lambda))=0\) for \(r'\) satisfying the condition \(\hat{\beta}(r', \lambda) \cdot \hat{e}_u(\lambda) > 0\).

For a point \(r' \in \mathbb{R}^3\), we introduce an extended data function

\[
\hat{D}(r_0(\lambda), \hat{\beta}(r', \lambda)) = D(r_0(\lambda), \hat{\beta}(r', \lambda)) - D(r_0(\lambda), -\hat{\beta}(r', \lambda))
\]
and its generalized backprojection

\[
g(r') = \int_{\lambda_1}^{\lambda_2} \frac{d\lambda}{|r' - r_0(\lambda)|} \partial_\lambda \hat{D}(r_0(q), \hat{\beta}(r', \lambda)) \bigg|_{q=\lambda}.
\]

For the source trajectory specified in Eq. (3), we define an image function \(f_\beta(r)\) to be computed as

\[
f_\beta(r) = \int_{\mathbb{R}^3} d\nu K(r, r') g(r'),
\]
where \(r\) denotes a point on the entire PI-line specified by \(\lambda_1\) and \(\lambda_2\), the integral kernel is given by

\[
K(r, r') = \frac{1}{2\pi j} \int_{\mathbb{R}^3} d\nu \sgn[\nu \cdot \hat{e}_u] e^{2\pi i \nu \cdot (r-r')},
\]
the unit vector \(\hat{e}_u=[r_0(\lambda_2)-r_0(\lambda_1)]/[|r_0(\lambda_2)-r_0(\lambda_1)|]\) indicates the direction of the PI-line, and “sgn” denotes the signum. In the Appendix, we show that the substitution of Eqs. (4), (7), and (9) into Eq. (8) yields

\[
f_\beta(r) = \sum_{i \in PI(r, \nu)} \sgn[\nu \cdot \hat{e}_u] \sgn[\nu \cdot \frac{d\nu}{d\lambda}] W(r, \nu),
\]
where \(F(\nu)\) denotes the 3D Fourier transform of the true image \(f(r)\), the weighting function is given by

\[
W(r, \nu) = \sum_{i \in PI(r, \nu)} \sgn[\nu \cdot \hat{e}_u] \sgn[\nu \cdot \frac{d\nu}{d\lambda}]
\]
and \(PI(r, \nu)\) indicates the set of intersections of a plane, which orients along \(\nu\) and contains \(r\), with the helix segment specified by \(\lambda \in [\lambda_1, \lambda_2]\), which we refer to as the PI-helix segment, and the rotation angle \(\lambda_i\) labels the \(i\)th intersection in the set.

Clearly, in Eq. (10), when the weighting function \(W(r, \nu)=1\) and \(0\) for \(r\) on the portions of the PI-line within and outside the helix cylinder, respectively, the function \(f_\beta(r)\) becomes the true image function \(f(r)\). It can be demonstrated that, for a helical scan with a constant pitch (i.e., for \(Z'(\lambda)=h/2\pi\)), the weighting function \(W(r, \nu)=1\) and \(0\) for \(r\) on the portions of the PI-line within and outside the helix cylinder, respectively. Below, we show that, under the condition in Eq. (12) below, \(W(r, \nu)\) remains unity and zero for \(r\) on the portions of the PI-line within and outside the helix cylinder, respectively, and thus that Eq. (8) provides a mathematically exact reconstruction formula for helical cone-beam CT with a variable pitch.

B. Evaluation of the weighting function

The value of the weighting function \(W(r, \nu)\) relies critically upon the properties of the trajectory projection onto the detector plane. It has been shown that, under the condition
\[ Z'(\lambda) + Z''(\lambda) > 0, \]  

(12)

the upper and lower boundaries of the Tam–Danielsson window remain concave and convex, respectively. Below, we prove that, when the upper and lower boundaries of the Tam–Danielsson window are concave and convex, respectively, [i.e., under the condition in Eq. (12)], the weighting function in Eq. (11) becomes unity and zero for \( \mathbf{r} \) on the portions of a PI-line within and outside the helix cylinder, respectively.

As mentioned above, \( \Pi(\mathbf{r}, \mathbf{v}) \) indicates the set of intersections of a plane, which has an orientation along \( \mathbf{v} \) and contains \( \mathbf{r} \), with the PI-helix segment, and the rotation angle \( \lambda_i \) labels the \( i \)th intersection in the set. When the point \( \mathbf{r} \) is on the PI-line segment (the portion of a PI-line within the helix cylinder), it can be shown that planes passing through \( \mathbf{r} \) can intersect with the PI-helix segment once, twice, and three times. However, the plane intersecting with the PI-helix segment twice does not contribute to image reconstruction. Conversely, when the point \( \mathbf{r} \) is on the portion of a PI-line outside the helix cylinder, it can be shown that plane passing through \( \mathbf{r} \) can intersect with the PI-helix segment once or twice. In this situation, one can readily show17 that the planes intersecting with the PI-helix segment twice contribute to image reconstruction.

For a PI-helix segment satisfying the condition in Eq. (12) and specified by \( \lambda \in [\lambda_1, \lambda_2] \), the weight function \( W(\mathbf{r}, \mathbf{v}) \) can be expressed in terms of the projections of \( \mathbf{v} \), \( \mathbf{d}_o(\lambda)/d\lambda \), and \( \hat{\mathbf{e}}(\lambda) \) onto the detector plane. Let \( \hat{\mathbf{e}}(\lambda) = \mathbf{d}_o(\lambda)/d\lambda \cdot |\mathbf{d}_o(\lambda)/d\lambda|^{-1} \) denote the unit vector along the tangential direction of the trajectory at rotation angle \( \lambda \), then one can readily show that \( \hat{\mathbf{e}}(\lambda) \) is parallel to the detector plane. It can also be shown that

\[ \hat{e}_\nu(\lambda) = \frac{1}{c_1}[\hat{e}_\pi(\lambda) \times \hat{\beta}] \times \hat{e}_\nu(\lambda) \quad \text{and} \quad \mathbf{v}_d(\lambda) = \hat{e}_\nu(\lambda) \times [\mathbf{v} \times \hat{e}_\nu(\lambda)], \]

(13)

where \( c_1 = |\hat{e}_\pi(\lambda) \times \hat{\beta}| \) is the normalization factor. It can be seen that Eq. (10) is mathematically equivalent to Eq. (A6) in which the \( \delta \)-function automatically imposes the condition \( \mathbf{v} \cdot \hat{\beta} = 0 \). Therefore, using this condition, we can obtain

\[ \text{sgn}[\hat{e}_\nu \cdot \mathbf{v}] = \text{sgn}[\hat{e}_\pi(\lambda) \cdot \mathbf{v}_d(\lambda)] \quad \text{and} \quad \text{sgn}[\mathbf{d}_o(\lambda)/d\lambda \cdot \mathbf{v}] = \text{sgn}[\hat{\mathbf{e}}(\lambda) \cdot \mathbf{v}_d(\lambda)]. \]

(14)

Substituting Eq. (14) into Eq. (11) yields

\[ W(\mathbf{r}, \mathbf{v}) = \sum_{i \in \Pi(\mathbf{r}, \mathbf{v})} \text{sgn}[\mathbf{v}_d(\lambda_i) \cdot \mathbf{e}_\nu(\lambda_i)] \cdot \text{sgn}[\mathbf{v}_d(\lambda_i) \cdot \hat{\mathbf{e}}(\lambda_i)]. \]

(15)

As shown in Fig. 1, at a rotation angle \( \lambda_0 \in [\lambda_1, \lambda_2] \), we use \( \Gamma_\ell \) and \( \Gamma_u \) to indicate the lower and upper boundaries of the Tam–Danielsson window, respectively. The solid sections on \( \Gamma_\ell \) and \( \Gamma_u \) correspond to the projections of the portions of the PI-helix segment specified by \( \lambda \in [\lambda_1, \lambda_0] \) and \( \lambda \in [\lambda_0, \lambda_2] \), respectively. Line \( \mathbf{L}_\pi \) shows the projection of the PI-line and passes through \( \mathbf{r}_d \). The asymptote \( \mathbf{L}_\alpha \) of the boundaries is determined by

\[ u = \frac{Z'(\lambda_0)}{R} \left[ u + \frac{Z''(\lambda_0)}{R} \right]. \]

(16)

It should be noted that line \( \mathbf{L}_\alpha \), in general, does not pass through the origin of the detector plane because \( Z'(\lambda_0) \neq 0 \).

We first consider the case in which \( \mathbf{r} \) is on the PI-line segment (i.e., it is on the portion of the PI-line within the helix cylinder). At a rotation angle \( \lambda_0 \in [\lambda_1, \lambda_2] \), let \( \mathbf{r}_d \) denote the cone-beam projection, onto the detector plane, of point \( \mathbf{r} \), and let line \( \mathbf{L} \) indicate the intersection between the detector and plane \( \Pi(\mathbf{r}, \mathbf{v}) \). As shown in Fig. 1(a), \( \mathbf{r}_d \) is within the
Tam–Danielsson window, and line \( L \) passes through point \( r_d \) because \( \Pi(r, \nu) \) contains points \( r \) and \( \lambda_0 \). We now define line \( L_0' \) that is parallel to \( L_0 \) and passes through point \( r_d \). We also use \( L_\nu \) to denote the line that passes through point \( r_d \) and is tangential to one of the solid portions on the boundaries. It can be observed that line \( L_\nu \) will be tangential to the solid portion on the upper or lower boundary if point \( r_d \) is above or below line \( L_\nu \). In summary, as shown in Fig. 1(a), lines \( L_t, L_\nu, \) and \( L_\nu' \) pass through the same point \( r_d \); and three regions \( D^{(1)}, D^{(3)}, \) and \( D^{(3)} \) are formed by \( L_t, L_\nu, \) and \( L_\nu' \), respectively.

Now, if line \( L \) is in \( D^{(1)} \), it will not intersect with the solid portions on the boundaries, suggesting that plane \( \Pi(r, \nu) \) intersects with the PI-helix segment only once at the source point \( \lambda_0 \). On the other hand, because of the concave and convex properties of the boundaries, when line \( L \) is in \( D^{(3)} \) or in \( D^{(3)} \), it intersects twice with the solid portions of the boundaries, indicating that plane \( \Pi(r, \nu) \) intersects with the PI-helix segment three times, including the one at \( \lambda_0 \). As Eq. (15) shows, the key for computing the weighting function lies in the evaluation of the product of signums, \( \text{sgn}[\nu_d(\lambda_1) \cdot \hat{e}_\nu(\lambda_1)] \) \( \text{sgn}[\nu_d(\lambda_3) \cdot \hat{e}_\nu(\lambda_3)] \). Considering the property of line \( L \) in \( D^{(1)}, D^{(3)}, \) and \( D^{(3)} \), and using the same phase convention in Refs. 11 and 17, we conclude that the signum is 1 when line \( L \) is in \( D^{(1)} \) or in \( D^{(3)} \) and −1 when line \( L \) is in \( D^{(3)} \), respectively.

When plane \( \Pi(r, \nu) \) intersects with the PI-helix segment only once, Eq. (15) contains only one term, and line \( L \) is in \( D^{(1)} \). Therefore,

\[
W(r, \nu) = \text{sgn}[\nu_d(\lambda_1) \cdot \hat{e}_\nu(\lambda_1)] \text{sgn}[\nu_d(\lambda_3) \cdot \hat{e}_\nu(\lambda_3)] = 1. \tag{17}
\]

When plane \( \Pi(r, \nu) \) intersects with the PI-helix segment three times, there are three terms in Eq. (15) determined, respectively, by \( \lambda_1, \lambda_3 \), and \( \lambda_3 \). At rotation angle \( \lambda_1 \), the other two intersections \( \lambda_2 \) and \( \lambda_3 \) are on the upper boundary \( \Gamma_u \), and line \( L \) is in \( D^{(3)} \). Therefore,

\[
\text{sgn}[\nu_d(\lambda_1) \cdot \hat{e}_\nu(\lambda_1)] \text{sgn}[\nu_d(\lambda_3) \cdot \hat{e}_\nu(\lambda_3)] = 1. \tag{18}
\]

At rotation angle \( \lambda_2 \), intersections \( \lambda_2 \) and \( \lambda_3 \) are on the lower boundary \( \Gamma_l \) and the upper boundary \( \Gamma_u \), respectively. In this case, line \( L \) is in \( D^{(3)} \), and thus

\[
\text{sgn}[\nu_d(\lambda_2) \cdot \hat{e}_\nu(\lambda_2)] \text{sgn}[\nu_d(\lambda_3) \cdot \hat{e}_\nu(\lambda_3)] = -1. \tag{19}
\]

Finally, at rotation angle \( \lambda_3 \), the other two intersections \( \lambda_1 \) and \( \lambda_3 \) are on the lower boundary \( \Gamma_l \), and thus line \( L \) is in \( D^{(3)} \). Therefore,

\[
\text{sgn}[\nu_d(\lambda_3) \cdot \hat{e}_\nu(\lambda_3)] \text{sgn}[\nu_d(\lambda_3) \cdot \hat{e}_\nu(\lambda_3)] = 1. \tag{20}
\]

Note that point \( r_d \) is below line \( L_\nu \) in this case.

Using Eqs. (18)–(20) into Eq. (15), we obtain

\[
W(r, \nu) = 3 \sum_{i=1}^{3} \text{sgn}[\nu \cdot \hat{e}_\nu(\lambda_i)] \frac{dr_\nu(\lambda_i)}{d\lambda} = 1, \tag{21}
\]

for a given point \( r \) on the portion of PI-line within the helix cylinder.

We then consider the case in which \( r \) is on the portion of the PI-line outside the helix cylinder. At a rotation angle \( \lambda_0 \in [\lambda_1, \lambda_3] \), let \( r_d \) denote the cone-beam projection, onto the detector plane, of point \( r \), and let line \( L \) indicate the intersection between the detector and plane \( \Pi(r, \nu) \) As discussed above, in this situation, one needs to evaluate \( W(r, \nu) \) only for planes that intersect with the PI-helix segment twice (or, equivalently, only for lines \( L \) intersect once with the solid portions of \( \Gamma_l \) or \( \Gamma_u \)). As displayed in Fig. 1(b), \( r_d \) passes through the Tam–Danielsson window, and line \( L \) passes through point \( r_d \) because \( \Pi(r, \nu) \) contains points \( r \) and \( \lambda_0 \). Let line \( L_\nu \) that is parallel to \( L \) passes through point \( r_d \). We can then define two regions \( D^{(2)} \) and \( D^{(3)} \), which are formed by \( L_\nu \) and \( L_\nu' \), and by \( L_\nu \) and \( L_\nu' \), respectively, on the detector plane. Following the phase convention discussed above, \( 17,11 \) one can conclude that the signum is 1 if line \( L \) is in \( D^{(2)} \) and −1 if line \( L \) is in \( D^{(2)} \), respectively.

When plane \( \Pi(r, \nu) \) intersects with the PI-helix segment twice, there are two terms in Eq. (15) determined by \( \lambda_1 \) and \( \lambda_2 \), respectively. At rotation angle \( \lambda_1 \), intersection \( \lambda_1 \) is on the upper boundary \( \Gamma_u \), and line \( L \) is in \( D^{(2)} \). Therefore,

\[
\text{sgn}[\nu_d(\lambda_1) \cdot \hat{e}_\nu(\lambda_1)] \text{sgn}[\nu_d(\lambda_2) \cdot \hat{e}_\nu(\lambda_2)] = 1. \tag{22}
\]

On the other hand, at rotation angle \( \lambda_2 \), intersection \( \lambda_2 \) is on the lower boundary \( \Gamma_l \). Therefore, line \( L \) is in \( D^{(2)} \), and

\[
\text{sgn}[\nu_d(\lambda_2) \cdot \hat{e}_\nu(\lambda_2)] \text{sgn}[\nu_d(\lambda_2) \cdot \hat{e}_\nu(\lambda_2)] = -1. \tag{23}
\]

Using Eqs. (22) and (23) into Eq. (15), we obtain

\[
W(r, \nu) = \sum_{i=1}^{3} \text{sgn}[\nu \cdot \hat{e}_\nu] \frac{d\nu(\lambda_i)}{d\lambda} = 0, \tag{24}
\]

for \( r \) on the portion of PI-line outside the helix cylinder.

Based upon the results in Eqs. (8), (21), and (24), we conclude that, for a helical trajectory with a variable pitch satisfying the condition in Eq. (12)

\[
f_d(r) = f(r) \tag{25}
\]

for \( r \) on the entire PI-line. Therefore, the combination of Eqs. (8) and (9) provides a general formula for exact image reconstruction on a PI-line from data acquired with a helical scan with the variable pitch.

C. Reconstruction algorithms

As demonstrated above, for a helical cone-beam scan with a variable pitch satisfying the condition in Eq. (12), \( f_d(r) \) in the general formula of Eq. (8) becomes the true image \( f(r) \). Therefore, one can use Eqs. (8) and (9) to derive algorithms for image reconstruction on PI-line segments.\(^{11,12}\) Two algorithms, i.e., the BPF and FBP algorithms, have been derived previously for image reconstruction in the case of a helical scan with a constant pitch. Following the same strategy,\(^{11,12}\) we can readily obtain the BPF and FBP algorithms for a helical cone-beam CT with a variable pitch satisfying Eq. (12). Let \( (\chi_\pi, \lambda_1, \lambda_2) \) denote the PI-line coordinates, where \( \lambda_1 \) and \( \lambda_2 \) identify the PI-line segment and \( \chi_\pi \) specifies a par-
ticular point on the PI-line segment. It can readily be shown that, for a given point on the PI-line segment, its fixed coordinates \((x, y, z)\) and PI-line coordinates \((x, \lambda_1, \lambda_2)\) are related through

\[
x = R[(1 - t(x))\cos \lambda_1 + t(x)\cos \lambda_2],
\]

\[
y = R[(1 - t(x))\sin \lambda_1 + t(x)\sin \lambda_2],
\]

\[
z = \frac{h}{2}\pi[(1 - t(x))\lambda_1 + t(x)\lambda_2],
\]

where

\[
t(x) = \frac{1}{2} + \frac{x}{r_0(\lambda_1) - r_0(\lambda_2)}. \tag{27}
\]

We use \(f(x, \lambda_1, \lambda_2)\) and \(g(x, \lambda_1, \lambda_2)\) to denote the image and the backprojection image [see Eq. (7)] in terms of the PI-line coordinates, respectively, which satisfy

\[
f(x, y, z) = f_{\pi}(x, \lambda_1, \lambda_2),
\]

\[
g(x, y, z) = g_{\pi}(x, \lambda_1, \lambda_2), \tag{28}
\]

under the condition that \((x, y, z)\) and \((x, \lambda_1, \lambda_2)\) are related through Eqs. (26) and (27).

1. The BPF algorithm

Let \(x_{\pi_1}\) and \(x_{\pi_2}\) denote the coordinates of the two ends of a PI-line segment specified by \(\lambda_1\) and \(\lambda_2\), and they can be determined by use of \(t = 0\) and \(t = 1\), respectively, in Eq. (27). Because the support of the image is confined entirely within the helix, we have

\[
f_{\pi}(x, \lambda_1, \lambda_2) = 0 \quad \text{for} \quad x \notin [x_{\pi_1}, x_{\pi_2}]. \tag{29}
\]

Using Eqs. (8) and (9), it can be shown

\[
g_{\pi}(x', \lambda_1, \lambda_2) = 2 \int_{x_{\pi_1}}^{x_{\pi_2}} \frac{dx}{x_{\pi}} f_{\pi}(x, \lambda_1, \lambda_2), \tag{30}
\]

where \(x' \in [x_{\pi_1}, x_{\pi_2}]\). Equation (30) is a finite Hilbert transform of the object function on the PI-line segment. The task of image reconstruction thus becomes the inversion of this finite Hilbert transform, which is well-known and is given by

\[
f_{\pi}(x, \lambda_1, \lambda_2) = \frac{1}{2\pi^2} \frac{1}{\sqrt{(x_{\pi_2} - x)(x_{\pi_2} - x_{\pi_1})}} \times \left[ \int_{x_{\pi_1}}^{x_{\pi_2}} \frac{dx}{x_{\pi}} \sqrt{(x_{\pi_1} - x')(x_{\pi_1} - x_{\pi_2})} \frac{g_{\pi}(x', \lambda_1, \lambda_2) + 2\pi D(r_0(\lambda_1), \xi_{\pi})}{x_{\pi} - x'} \right], \tag{31}
\]

where \(x' \in [x_{\pi_1}, x_{\pi_2}]\). The second term in the square bracket on the right-hand side of Eq. (31) was determined from the integration of the object function over the PI-line segment and thus can be obtained directly from the data \(D(r_0(\lambda_1), \xi_{\pi}), \xi_{\pi}\). We refer to Eq. (31) as the BPF algorithm for helical cone-beam CT with a variable pitch. Similar to the previous BPF algorithms, the BPF algorithm is capable of reconstructing exactly region-of-interest images from data containing transverse truncations.

Let \(P(u, v, \lambda)\) denote the physical data in terms of the coordinates \(u\) and \(v\) on the detector plane, and thus

\[
P(u, v, \lambda) = D(r_0(\lambda), \hat{\beta}), \tag{32}
\]

provided that

\[
\hat{\beta} A(u, v) = u \hat{\epsilon}_u(\lambda) + v \hat{\epsilon}_v(\lambda) - S \hat{\epsilon}_s(\lambda) \quad \text{and} \quad A(u, v) = \sqrt{u^2 + v^2 + S^2}, \tag{33}
\]

where \(u\) and \(v\) denote, as shown in Fig. 1, the coordinates of the point at which the ray intersects with the detector plane.

Using Eqs. (32) and (33) the data derivative with respect to the rotation angle can be rewritten as that with respect to coordinates \(u\) and \(v\) on the detector plane. Substituting these results in Eq. (7) yields

\[
g_{\pi}(x', \lambda_1, \lambda_2) = \int \frac{d\lambda}{\lambda_1} \frac{P'(u', v', \lambda)}{|r - r_0(\lambda)|} + \frac{P(u', v', \lambda)}{|r - r_0(\lambda)|}; \tag{34}
\]

where, for given \((x', \lambda_1, \lambda_2)\), \(u'\) and \(v'\) are determined by use of Eqs. (1), (2), and (26), and

\[
\text{Fig. 2. Images} f_{\pi}(x, \lambda_1, \lambda_2) \text{ (left column) reconstructed by use of the BPF (upper row) and the FBP (lower row) algorithms, on PI-line segments specified by} \lambda_1 = \pi \text{ and} \lambda_2 \in [-0.14\pi, 0.14\pi]. \text{ The set of} \lambda_2 \text{ forms the vertical axis, whereas} x_\pi \text{, the coordinate on the PI-line segment, forms the horizontal axis (in unit of cm). The display window is} [1.0, 1.0, 0.5]. \text{ The panels in the right column show the profiles of} f_{\pi}(x, \lambda_1, \lambda_2) \text{ on a PI-line segment specified by} \lambda_1 = \pi \text{ and} \lambda_2 = -0.02\pi \text{, obtained by use of the BPF (upper) and FBP (lower) algorithms, respectively.}
\]
\[
P'(u,v,\lambda) = -\left[ \frac{d\mathbf{r}_0(\lambda)}{d\lambda} \cdot \hat{B} \right] P(u,v,\lambda) \\
+ \left[ \frac{d\mathbf{r}_0(\lambda)}{d\lambda} \cdot \hat{e}_y(\lambda) \right] A(u,v) \frac{\partial P(u,v,\lambda)}{\partial u} \\
+ \left[ \frac{d\mathbf{r}_0(\lambda)}{d\lambda} \cdot \hat{e}_z(\lambda) \right] A(u,v) \frac{\partial P(u,v,\lambda)}{\partial v}. \\
\]

The expressions in Eqs. (34) and (35) were used in our numerical study in Sec. III below.

2. The FBP algorithm

From Eqs. (8) and (9), following the strategy in Ref. 12, we can also derive an FBP algorithm for image reconstruction on a PI-line segment specified by \(\lambda_1\) and \(\lambda_2\). Moreover, it has also been shown in Ref. 16 that the second term of the extended data function in Eq. (6) does not contribute to the FBP reconstruction for points \(r\) on a PI-line segment. Consequently, in this situation, only the physical data \(D(\mathbf{r}_0(\lambda), \hat{\mathbf{B}}(\mathbf{r}', \lambda))\) in Eq. (4) contributes to the backprojection image \(g_\pi(x', \lambda_1, \lambda_2)\) in Eq. (7).

Similar to this situation for the BPF algorithm, using the physical data \(P(u,v,\lambda)\) in terms of the coordinates \(u\) and \(v\) on the detector plane, and re-expressing its derivative with respect to the rotation angle as the derivative with respect to \(u\) and \(v\), the FBP reconstruction on the PI-line segment can be written as

\[
f_\pi(x, \lambda_1, \lambda_2) = \frac{1}{2\pi} \int_{\lambda_1}^{\lambda_2} d\lambda \int_{\mathbb{R}} \frac{A^2(u,v)}{|\mathbf{r} - \mathbf{r}_0(\lambda)|^2} \left| \frac{d^2 u'}{u'' - u'''} A^2(u',v') \right| \frac{A(u,v)}{|\mathbf{r} - \mathbf{r}_0(\lambda)|} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{A(u,v)}{|\mathbf{r} - \mathbf{r}_0(\lambda)|} \int_{\mathbb{R}} \frac{A(u,v)}{u'' - u'''} A(u',v') \right|_{\lambda_1}^{\lambda_2},
\]

where \(P'(u,v,\lambda)\) is given by Eq. (35).

The above discussion indicates that, under the condition in Eq. (12), both BPF and FBP algorithms reconstruct exactly the image on an individual PI-line segment. Furthermore, it has been shown previously\(^{15}\) that this condition also allows the helix cylinder to be filled completely by nonintersecting PI-line segments. Therefore, one can reconstruct exactly the image within the helix cylinder through the image reconstruction on these nonintersecting PI-line segments by use of the proposed BPF or FBP algorithm.

III. NUMERICAL RESULTS

We have performed a preliminary numerical study to quantitatively validate and demonstrate the proposed BPF and FBP algorithms for a helical cone-beam scan with \(Z(\lambda) = (h_0/2\pi)^{\lambda}\), where \(h_0 > 0\) is a constant. It can readily be verified that this function satisfies the condition in Eq. (12). The radius of the helical trajectory is \(R = 570\) mm, and \(h_0 = 10\) mm/radian\(^3\). Using this configuration, we generated cone-beam data from the Shepp–Logan phantom at 1024 rotation angles uniformly distributed between \(\lambda_{\text{min}} = -\pi\) and \(\lambda_{\text{max}} = \pi\). Parameters \(\lambda_{\text{min}}, \lambda_{\text{max}},\) and \(h_0\) were so chosen to form a trajectory that covers only the central portion of the Shepp–Logan phantom, simulating a long object problem.

![Images](image1.png)

**Fig. 3.** Images \(f_\pi(x, \lambda_1, \lambda_2)\) reconstructed, by use of the BPF (upper row) and the FBP (lower row) algorithms, on PI-line segments specified by (a) \(\lambda_1 = -0.84\pi\) and \(\lambda_2 \in [0.02\pi, 0.30\pi]\), (b) \(\lambda_1 = -0.24\pi\) and \(\lambda_2 \in [0.62\pi, 0.90\pi]\), and (c) \(\lambda_1 = -0.13\pi\) and \(\lambda_2 \in [0.73\pi, 1.0\pi]\), respectively. Again, the sets of \(\lambda_s\) form the vertical axes, whereas \(s_y\), the coordinates on the PI-line segments, form the horizontal axes (in unit of cm). The display window is \([1.0, 1.05]\).

![Images](image2.png)

**Fig. 4.** Images obtained by use of the BPF (upper row) and FBP (lower row) algorithms at 2D slices specified by (a) \(x = 0\) cm, (b) \(y = -2.7\) cm, and (c) \(z = 0\) cm, respectively. The display window is \([1.0, 1.05]\).
virtual flat panel detector, consisting of $512 \times 512$ detection elements, locates at the center of rotation. Therefore, the distance between the source and the detector is $S = R$. The size of the detection element is $0.5 \times 0.5$ mm$^2$.

From the generated data, we reconstructed images on PI-line segments by use of the proposed BPF and FBP algorithms in Sec. II C. In Fig. 2, we display images (left) and the corresponding profiles (right) on PI-line segments reconstructed by use of the BPF and FBP algorithms. In particular, the images were obtained for a set of PI-lines specified by $\lambda_1 = -\pi$ and $\lambda_2 \in [-0.14\pi, 0.14\pi]$, and the collection of $\lambda_2$ forms the vertical axis, whereas $x_m$ the coordinate on the PI-line segments, forms the horizontal axis (in unit of cm). The images on PI-line segments appear visually “distorted,” because they are not displayed in the fixed-coordinate system. However, the image values on each PI-line segment, as the profiles show, are virtually identical to the exact values. The images on PI-line segments obtained by use of the BPF and FBP algorithms are also visually different in shapes. This is because we have used different numbers of points on PI-line segments for image reconstruction in each case. However, the profiles obtained with the two algorithms that are displayed in physical length units are virtually identical. In Fig. 3, we show additional images on PI-line segments reconstructed by use of the BPF (upper row) and FBP (lower row) algorithms, respectively. Images in each row in Fig. 3 were obtained for sets of PI-line segments identified by (a) $\lambda_1 = -0.84\pi$ and $\lambda_2 \in [0.02\pi, 0.30\pi]$, (b) $\lambda_1 = -0.24\pi$ and $\lambda_2 \in [0.62\pi, 0.90\pi]$, and (c) $\lambda_1 = -0.13\pi$ and $\lambda_2 \in [0.73\pi, 1.0\pi]$, respectively.

Using the relationship between $(x, y, z)$ and $(x_m, \lambda_1, \lambda_2)$ in Eqs. (26) and (27), one can convert the image $f(x_m, \lambda_1, \lambda_2)$ on PI-line coordinates into an image $f(x, y, z)$ on the fixed coordinates. In this work, such a conversion was accomplished by use of a simple 3D linear interpolation scheme. We show 2D slices at (a) $x = 0$ cm, (b) $y = -2.7$ cm, and (c) $z = 0$ cm, respectively, in the converted 3D images in Fig. 4. Again, the upper and lower rows were obtained from images on PI-line segments reconstructed by use of the BPF and FBP algorithms. We also display in Fig. 5 the corresponding profiles in images shown in Fig. 4. The profiles were on lines identified by (a) $x = 0$ cm and $z = 0.8$ cm, (b) $x = 0$ cm and $y = -2.7$ cm, and (c) $y = -2.2$ cm and $z = 0$ cm, respectively, obtained by use of the BPF (upper row) and FBP (lower row) algorithms. We also use dashed lines to show the corresponding true profiles. Furthermore, we display in Fig. 6 images obtained by use of the BPF (upper row) and FBP (lower row) algorithms at additional transverse slices that are specified by (a) $z = -1.5$ cm, (b) $z = 1.1$ cm, and (c) $z = 1.6$ cm, respectively.

Finally, we demonstrate the image reconstruction from noisy data acquired by use of a helical cone-beam scan with a variable pitch. Using the noiseless data, which were employed to produce the images in Figs. 4 and 6, as the mean values, we generate noisy data from a Gaussian with a standard deviation that is 0.1% of the maximum value in the noiseless data. From such noisy data, we use the proposed
BPF and FBP algorithms to reconstruct images, which are displayed in Figs. 7 and 8. Furthermore, we generated data containing Poisson noise by assuming that the incident flux of the projection is about 10^6. From such data containing Poisson noise, we reconstructed images by use of the proposed BPF and FBP algorithms and display them in Fig. 9.

IV. CONCLUSION

We have previously developed a general formula and, based upon which, we have derived two algorithms, i.e., the BPF and FBP algorithms, for exact image reconstruction on PI-line segments in the case of a helical cone-beam scan with a constant pitch. In this work, we have extended this formula, and the BPF and FBP algorithms as well, to address the problem of image reconstruction in the situation of a helical cone-beam scan with a variable pitch. We have shown that, under the condition in Eq. (12), the formula remains valid. We have subsequently generalized the BPF and FBP algorithms to reconstruct exactly images on PI-line segments from data acquired by use of a helical cone-beam scan with such a variable pitch. Other investigators have identified a sufficient condition on PI-lines under which the nonintersecting PI-line segments can completely cover the volume confined by a helix with a variable pitch. This PI-line condition is a special case of the condition in Eq. (12). The image within the helix cylinder can be obtained through the image reconstruction on the nonintersecting PI-line segments, which can fill completely the helix cylinder, by use of the proposed BPF and FBP algorithms. We have performed a preliminary numerical study to quantitatively demonstrate and validate the proposed algorithms, and the results in the study confirm that our algorithms can accurately reconstruct images from data acquired by use of a helical cone-beam scan with a variable pitch. The proposed algorithms in this work can readily be generalized to reconstruct exactly images within a region-of-interest (ROI) from data containing transverse truncations.

ACKNOWLEDGMENTS

This work was supported in part by National Institutes of Health Grant Nos. EB00225 and EB004287. Its contents are solely the responsibility of the authors and do not necessarily represent the official views of the National Institutes of Health. The authors gratefully acknowledge the use of the Chiba City Linux cluster in the Mathematics and Computer Science Division of Argonne National Laboratory.

APPENDIX:

We show below that Eqs. (8) and (10) are equivalent. Using Eqs. (4) and (6) into Eq. (8), one obtains
Substituting Eq. \( \mathcal{F} \) yields

\[
\frac{1}{10\pi}
\]

Using \( f(r) = \int d\mathbf{v} F(v) e^{i\mathbf{v} \cdot \mathbf{r}} \) in Eq. (A1) yields

\[
\frac{1}{10\pi}
\]

Replacing \( t \) by \( t' \) and \( \mathbf{r}_0 \) and using Eq. (5), we have

\[
\frac{1}{10\pi}
\]

Substituting Eq. (9) into Eq. (A3) yields

\[
\frac{1}{10\pi}
\]

Performing the integration over \( \mathbf{r}' \) in Eq. (A4), we obtain

\[
\frac{1}{10\pi}
\]

Carrying out the integrations over \( \mathbf{v}' \) and \( t' \) in Eq. (A5), we obtain

\[
\frac{1}{10\pi}
\]

The \( \delta \)-function in Eq. (A6) can be rewritten as

\[
\frac{1}{10\pi}
\]

where \( \{ \lambda_i^{(s)} \} \) are the solutions to \( \mathbf{v} \cdot \mathbf{r} - \mathbf{v} \cdot \mathbf{r}_0 = 0 \). Substituting Eq. (A7) into Eq. (A6), one has

\[
\frac{1}{10\pi}
\]

Clearly, Eq. (A8) is identical to (10). Therefore, Eqs. (8) and (10) are equivalent.


