General Formula for Fan-Beam Computed Tomography

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In this Letter, we derive a general reconstruction formula for fan-beam computed tomography (CT) utilizing the two-dimensional Dirac function, which is useful in CT imaging for moving objects.

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In 1972, about 80 years after Roentgen discovered x rays, the British engineer Hounsfield invented the first x-ray computed tomography (CT) scanner, which allows physicians to view internal organs noninvasively and scientists to evaluate compound materials nondestructively [1]. For this great invention, he shared a Nobel Prize in Medicine in 1979 with an American physicist Cormack, who contributed to the CT theory [2,3].

The fundamental principle of x-ray CT is to reconstruct an object from its known projections [1]. Suppose that the spatial distribution of x-ray attenuation coefficients of an object is μ(⃗r). According to the Lambert-Beers law [1], for a thin monoenergetic x-ray beam passing through the object, the incident and transmitted x-ray intensities I0 and I are related by $I = I_0 \exp\left(-\int_{\Gamma} \mu(\vec{r})dl\right)$, where the line integral is along the x-ray path Γ. Since the incident and transmitted x-ray intensities are measured, the line integral of the attenuation coefficients is known via $\int_{\Gamma} \mu(\vec{r})dl = \log(I_0/I) = P$. The goal of CT is to reconstruct μ(⃗r) from the known line integrals or projections $P$ using an appropriate formula.

X-ray photons emitting from an x-ray tube form a divergent fan beam in a two-dimensional space. For projection measurement, the x-ray source moves along a trajectory around the object. For a circular trajectory, the reconstruction formula was developed based on Randon’s formula [4,5]. Although significant efforts were made to generalize this result to other trajectories, it is difficult to remove certain constraints on the locus [6–11].

In this Letter, we propose a versatile and concise fan-beam reconstruction formula for a general scanning trajectory. The key step in our derivation is to express the two-dimensional Dirac function in a new integral form. We pay attention to the closed locus since the open locus can be closed by a virtual path. An exemplary application of this formula is described for CT imaging of moving objects.

Reconstruction formula for a closed locus.—First, let us define a turn number function for a general closed locus and formulate an expression of the two-dimensional Dirac function. Then a general relation between a function and its fan-beam projection will appear in a natural way.

As shown in Fig. 1, $C$ is an oriented closed curve in the $OXY$ plane. We define a turn number function (TNF) $N(\vec{r}) = n$ (1) as the number of times the curve $C$ goes counterclockwise around a given point $\vec{r}$. For points $O$, $a$, $b$, and $c$, we have: $N(O) = 2$, $N(a) = -1$, $N(b) = 1$, $N(c) = 0$.

Clearly, the TNF is undefined for points on the curve, which forms a point set of measure 0.

Let us start with the two-dimensional Dirac function

$$\delta(\vec{r}) = \int_0^{2\pi} \int_0^\infty \exp(2\pi i\omega(x\cos\theta + y\sin\theta))\omega dw d\theta$$

$$= \int_0^\pi h(x\cos\theta + y\sin\theta)d\theta,$$ (2)

where the function $h(t) = \int_{-\infty}^{\infty} |\omega| \exp(i2\pi\omega t) d\omega$ is called a ramp filter in the field of CT [1]. An intuitive interpretation of Eq. (2) can be found in Ref. [12]. Here $\vec{r} = (x, y)$ is any point in a two-dimensional plane.

As shown in Fig. 2, $S$ is a point on the closed curve $C$ in the $OXY$ plane. $\vec{O}$ is a point in the plane but not on the curve, with a position vector $\vec{r}_0 = (x_0, y_0)$. $\vec{O} \hat{Y}$ is parallel to the axis $OY$, and $\angle \vec{Y} \hat{O} \vec{S} = \beta$.

FIG. 1. The turn number function for an oriented closed curve $C$. 

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T is a point in the plane with a position vector \( \vec{r} = (x, y) \). Based on Eq. (2), we have

\[
\frac{1}{2} \int_C h((x - x_0) \cos \beta + (y - y_0) \sin \beta) d\beta = \frac{1}{2} \int_0^{2\pi} N(\vec{r}_0) h((x - x_0) \cos \beta + (y - y_0) \sin \beta) d\beta = N(\vec{r}_0) \delta(\vec{r} - \vec{r}_0), \tag{3}
\]

where \( \int_C \) represents the integral along the oriented curve \( C \).

Denoting \( ST = L \) and \( \angle OST = \tilde{\gamma} \), we have

\[
(x - x_0) \cos \beta + (y - y_0) \sin \beta = L \sin \tilde{\gamma}.
\]

Therefore, Eq. (3) becomes

\[
\frac{1}{2} \int_C h(L \sin \tilde{\gamma}) d\tilde{\beta} = N(\vec{r}_0) \delta(\vec{r} - \vec{r}_0). \tag{4}
\]

Given

\[
\angle OS\vec{O} = \gamma_0, \quad S\vec{O} = L_0, \quad S\vec{O} = D,
\]

\[
\angle OST = \gamma, \quad \angle YOS = \beta,
\]

we have

\[
d\tilde{\beta} = \frac{1}{L_0} (D \cos \gamma_0 d\beta - \sin \gamma_0 d\theta),
\]

as illustrated in Fig. 3. Therefore, Eq. (4) can be rewritten as

\[
\frac{1}{2} \int_C h(L \sin(\gamma - \gamma_0)) \times \frac{1}{L_0} (D \cos \gamma_0 d\beta - \sin \gamma_0 d\theta) = N(\vec{r}_0) \delta(\vec{r} - \vec{r}_0). \tag{5}
\]

Thus, the Dirac function has been expressed as a superposition (integration) of many fan-beam filters.

Suppose \( \Psi(\vec{r}_0) \) is a two-dimensional test function [13]. Multiplying Eq. (5) by the test function and integrating the two sides of (5) in the two-dimensional space \( XOY \), we obtain

\[
\frac{1}{2} \int_C \int_{\mathbb{R}^2} h(L \sin(\gamma - \gamma_0)) \frac{1}{L_0} (D \cos \gamma_0 d\beta - \sin \gamma_0 d\theta) \times \Psi(\vec{r}_0) d^2\vec{r}_0 = \int_{\mathbb{R}^2} N(\vec{r}_0) \delta(\vec{r} - \vec{r}_0) \Psi(\vec{r}_0) d^2\vec{r}_0. \tag{6}
\]

Exchanging the order of the two integrals on the left-hand side, we arrive at

\[
\frac{1}{2} \int_C \int_{\mathbb{R}^2} \Psi(\vec{r}_0) h(L \sin(\gamma - \gamma_0)) \times \frac{1}{L_0} (D \cos \gamma_0 d^2\vec{r}_0 d\beta - \sin \gamma_0 d^2\vec{r}_0 d\theta) = N(\vec{r}) \Psi(\vec{r}). \tag{7}
\]

In Fig. 2, we consider \( SO \) as an axis of a polar coordinate system and \( S \) as the origin of the system. In the polar coordinate system, the area element is \( d^2\vec{r}_0 = L_0 dL_0 d\gamma_0 \). Hence, Eq. (7) becomes

\[
\frac{1}{2} \int_C \int_{\gamma_0}^{+\pi} \int_{-\pi}^{\pm\pi} \Psi(\vec{r}_0) h(L \sin(\gamma - \gamma_0)) \times (D \cos \gamma_0 d\gamma_0 d\beta - \sin \gamma_0 d\gamma_0 d\theta) = N(\vec{r}) \Psi(\vec{r}). \tag{8}
\]

Denoting the projection in Eq. (8) as

\[
\int_{\gamma_0}^{+\pi} \Psi(\vec{r}_0) dL_0 = \int_{\gamma_0}^{+\pi} \Psi(OS + L_0 \tilde{n}_\gamma) dL_0 = P_s(\gamma_0),
\]

where \( \tilde{n}_\gamma \) is the unit vector associated with \( \gamma_0 \), we have the final relation between the test function and its fan-beam projections:

\[
\frac{1}{2} \int_C \int_{\gamma_0}^{+\pi} P_s(\gamma_0) h(L \sin(\gamma - \gamma_0)) \times (D \cos \gamma_0 d\gamma_0 d\beta - \sin \gamma_0 d\gamma_0 d\theta) = N(\vec{r}) \Psi(\vec{r}). \tag{9}
\]

This is a new fan-beam reconstruction formula for a general closed curve \( C \). The locus should be continuous but does not need to be differentiable [11].

Note that the value of a point on the curve \( C \) cannot be recovered by Eq. (9), since the TNF \( N(\vec{r}) \) is not defined on

FIG. 2. The geometry for fan-beam reconstruction with a general closed scanning curve.

FIG. 3. The relation between \( d\beta \) and \( d\tilde{\beta} \). \( SS' \) is a differential arc. From \( S \) to \( S' \), the position parameters \((D, \beta)\) are changed to \((D + dD, \beta + d\beta)\). This process can be divided into two steps: \( S \rightarrow M \) and \( M \rightarrow S' \) with \( OM = OS = D \).
the curve. However, the value of such points can be obtained by the continuity of the test function $\Psi(\tilde{r})$. For a point outside the curve $C$, its value cannot be recovered because $N(\tilde{r}) = 0$.

If the closed curve $C$ is described by two smooth or piecewise smooth functions

$$D = D(t), \quad \beta = \beta(t), \quad t \in [0, t_0],$$

Eq. (9) can be rewritten as

$$\frac{1}{2} \int_0^{t_0} \int_{-\pi}^{\pi} P(\gamma_0) h(L \sin(\gamma - \gamma_0)) \times (D \cos \gamma_0 \beta'(t) - \sin \gamma_0 D'(t)) d\gamma_0 dt = N(\tilde{r}) \Psi(\tilde{r}).$$

Here the symbol ($'$) stands for the first order derivative over variable $t$.

If the closed curve $C$ is described by a smooth or piecewise smooth function $D = D(\beta)$ with $\beta \in [0, 2\pi]$, Eq. (9) becomes

$$\frac{1}{2} \int_0^{2\pi} \int_{-\pi}^{\pi} P(\beta, \gamma_0) h(L \sin(\gamma - \gamma_0)) \times (D \cos \gamma_0 - \sin \gamma_0 D'(\beta)) d\gamma_0 d\beta = N(\tilde{r}) \Psi(\tilde{r}).$$

In the circular scanning case, $D = D_0 = \text{const}$, we return to the classical reconstruction formula [1,4,5]:

Numerical simulation.—As shown in Fig. 4, when a head is CT scanned for a $360^\circ$ angular range, an arbitrary head motion is equivalent to a general scanning trajectory around the head that is stationary. In principle, the head motion can be recorded by a camera. As a simple example, let us assume that the head motion is described by

$$x(t) = \frac{1}{2} \sin(\pi t / T), \quad y(t) = -\frac{1}{2} \cos(2\pi t / T),$$

$$\alpha(t) = -\frac{\pi}{2} \sin(2\pi t / T),$$

where $t \in [0, T]$, and $T$ is the time for a $360^\circ$ scan, $O(x, y)$ the center of the head, and $\alpha$ the rotational angle of the head. In the coordinate system $XOY$, the source moves along a circle ($\beta = \frac{2\pi}{T}$). The compound trajectory in the head system $O'X'Y'$ is shown in Fig. 5, which clearly violates the constraints in Refs. [6–9].

We used the Shepp-Logan phantom [1] to simulate the head and Eq. (9) to reconstruct the image. In the simulation, the following parameters were used: radius of the circular orbit $D_0 = 4$, 512 projections, and 2048 detector elements uniformly distributed along a half circle. The display window was set to $[0.99, 1.05]$. The ideal and reconstructed images are shown in Figs. 6(a) and 6(b), and the profiles of the midline plotted in Fig. 6(c). The reconstructed image was compared to the ground truth with an error 0.37% defined by

$$\text{error} = \sqrt{\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} (x_{i,j} - x'_{i,j})^2} / \sqrt{\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} x_{i,j}^2},$$

where $N = 256$, and $x_{i,j}$ and $x'_{i,j}$ are the original and reconstructed pixel values, respectively.

Discussion on the open locus and finite object case.—It is easy to see that Eq. (9) is valid for an open locus which has two asymptotes with opposite directions, such as a $U$-shaped locus [Fig. 7(a)]. If we allow the TNF to be an

![Fig. 4. The scanner coordinate system $OXY$ and the head coordinate system $O'X'Y'$.](image)

![Fig. 5. The compound source locus in the head coordinate system $O'X'Y'$.](image)
the straight line [Fig. 7(b)]. In Fig. 7(c), ACB is a general continuous open locus of a finite length. For a point \( \tilde{r} \) on the straight line AB, whose turn number may be an integer or an integer plus a half, we have

\[
\frac{1}{2} \int_{\overline{ACB}} \int_{-\pi}^{\pi} P_S(\gamma_0) h(L \sin(\gamma - \gamma_0))(D \cos \gamma_0 d\gamma_0 d\beta - \sin \gamma_0 d\gamma_0 dD) = \Psi(\tilde{r}),
\]

where \( P_A(\gamma_0) \) is the projection data and \( (L_A, \gamma_A) \) the polar coordinate of the reconstructed point \( \tilde{r} \) when the source appears at point A, and so on. To verify this, one can just apply Eq. (9) to the virtual locus IACBJ for a point \( \tilde{r} \) between A and B and the virtual locus ACBA for a point \( \tilde{r} \) beyond A and B.

Given a finite object surrounded by a continuous or piecewise continuous open locus \( C^* \) [in Fig. 7(d), \( C^* = AB + CD \)], if every straight line passing through the object intersects the open locus at least once, one can reconstruct it by closing the locus with virtual paths (BC and DA), since the projection data on the virtual paths can be obtained from the measured data. Alternatively, one can reconstruct the object by a weighted form of Eq. (9):

\[
\frac{1}{2} \int_{C} \int_{-\pi}^{\pi} w_S(\gamma_0) P_S(\gamma_0) h(L \sin(\gamma - \gamma_0)) (D \cos \gamma_0 d\gamma_0 d\beta - \sin \gamma_0 d\gamma_0 dD) = \Psi(\tilde{r}), \tag{11}
\]

where the weighting function \( w_S(\gamma_0) \) can make every line through the object contribute to the reconstructed image once and only once. If line \( a \) passing through the object intersects the open locus \( C^* \) \( n \) times, we require that \( \sum_{i=1}^{n} w_S(\gamma_{0a_i}) = 1 \), where \( \gamma_{0a} \) is the fan angle related to the line \( a \). For a finite object surrounded by a closed locus \( C \) (as a special case of open locus \( C^* \)) for \( N \) turns, we return to