Correspondence

Comment on “Image reconstruction via the finite Hilbert transform of the derivative of the backprojection” [Med. Phys. 34, 2837–2843 (2007)]

(Received 29 June 2007; revised 9 July 2007; accepted for publication 18 July 2007; published 26 September 2007)

To the Editor,

In a recent paper [Med. Phys. 34, 2837–2843 (2007)], Zeng presented a different version of the two-step Hilbert transform method for 2D image reconstruction by exchanging the order of the differentiation and backprojection. From the viewpoint of the numerical implementation, the differentiation in the image domain can be more desirable than in the projection domain. However, in Zeng’s paper the deduction from the parallel-beam formula to the fan-beam form is not rigorous. For clarity, we provide here a proof for Zeng’s formula and point out some alternative formulations.

The two-step Hilbert transform method for 2D image reconstruction developed by Noo et al.† can be expressed as

\[ f(\bar{x}) = f(x, y) = \frac{1}{2\pi} H_x \int_0^\pi \frac{\partial}{\partial s} p(s, \theta) \left|_{s=\bar{x}\theta} \right. d\theta \]

\[ = \frac{1}{2\pi} H_x b(x, y), \]

(1)

where \( f(x, y) \) is an object function, \( p(s, \theta) \) the parallel-beam projection, \( H_x \) the (finite) inverse Hilbert transform along the \( x \) direction, and \( b(x, y) \) the backprojection of the derivative of the parallel-beam projection.

\( \theta=\sin \theta, \cos \theta \) and \( \bar{x}=\overline{OT}(x, y) \) are the unit directional vector and positional vector in the 2D space, as shown in Fig. 1 in Ref. 2. For convenience, we keep the notations the same as in Ref. 2.

Zeng reported that the backprojected derivative image in Eq. (1) could be calculated in an alternative way: differentiation and then backprojection,

\[ b(x, y) = \frac{\partial}{\partial x} b_1(x, y) + \frac{\partial}{\partial y} b_2(x, y), \]

(2)

with

\[ b_1(x, y) = -\int_0^\pi p(s, \theta) \left|_{s=\bar{x}\theta} \right. \sin \theta d\theta \]

\[ b_2(x, y) = \int_0^\pi p(s, \theta) \left|_{s=\bar{x}\theta} \right. \cos \theta d\theta \]

(3)

Zeng further extended formula (3) to the case of fan-beam projections \( p_{\text{curve}}(\alpha, \beta) \):

\[ b_3(x, y) = -\frac{1}{2} \int_0^{2\pi} p_{\text{curve}}(\alpha_\xi, \beta) |\sin(\alpha_\xi + \beta)| d\beta \]

\[ b_3(x, y) = \frac{1}{2} \int_0^{2\pi} p_{\text{curve}}(\alpha_\xi, \beta) \cos((\alpha_\xi + \beta) \mod \pi) d\beta. \]

(4)

Coincidentally, we recently extended our SPIE conference paper‡ and developed a general two-step CT scheme,§ which led to two new fan-beam formulas, equivalent to an interchange of the order of the derivation and backprojection in Noo’s two formulas.‖ Based on our knowledge on the same issue, we have two immediate positive comments on Zeng’s paper: § (1) the methodology is applicable to Noo’s supper short scan formula§ as well; (2) Zeng’s formula is more stable in numerical implementation than ours due to the different positions of the sine factor.

However, the deduction in Ref. 2 from the parallel-beam formula Eq. (3) to the fan-beam form Eq. (4) is not rigorous. In fact \( d\beta=\theta d\theta \) holds only for \( \bar{x}=0 \) (Fig. 1). Hence, formulas (9) and (10) in Ref. 2 hold only for \( \bar{x}=0 \) (Fig. 2). For clarity, a proof for Zeng’s formula is outlined as follows.

From Eq. (3), a weighted form in the parallel-beam geometry can be expressed as

\[ b_3(x, y) = -\int_{\theta=0}^{\theta=2\pi} W(s, \theta)p(s, \theta) \left|_{s=\bar{x}\theta} \right. |\sin \theta| d\theta, \]

(5)

where a weight function must satisfy that \( W(s, \theta)+W(-s, \theta+\pi)=1 \) since the two rays parameterized by \( (s, \theta) \) and \((-s, \theta+\pi)\) are the same.

For a circular locus with radius \( R \), the relation between \( d\theta \) and \( d\beta \) is (see Fig. 1)

\[ d\theta = \frac{R \cos \alpha_\xi}{L} d\beta, \]

(6)

where \( L=|ST| \) is the distance between the source position \( S \) and reconstructed point \( T \). Therefore, formula (5) becomes

\[ b_3(x, y) = -\int_{\beta=0}^{\beta=2\pi} w(\alpha_\xi, \beta)p_{\text{curve}}(\alpha_\xi, \beta) \]

\[ \times |\sin(\alpha_\xi + \beta)| \frac{R \cos \alpha_\xi}{L} d\beta. \]

(7)

In the fan-beam geometry, the weight function should be normalized as

\[ w(\alpha_\xi, \beta)+w(-\alpha_\xi, \beta+\pi+2\alpha_\xi)=1, \]

since the rays parameterized by \( (\alpha_\xi, \beta) \) and \((-\alpha_\xi, \beta+\pi+2\alpha_\xi) \) are the same.
It is easy to verify that for any \( x \)
\[
T = \frac{\sin x}{H_20849} \frac{\pi}{H_11022} \]
the weight function in Eq. (7) becomes
\[
b_s(x, y) = -\frac{1}{2} \int_0^{2\pi} p_{\text{curve}}(x, \beta) |\sin(\alpha_\beta + \beta)| \frac{R \cos \alpha_\beta}{L} d\beta.
\]
Taking another weight function
\[
2 w(x, \beta) = \frac{L}{2 R \cos \alpha_\beta},
\]
Eq. (7) becomes
\[
b_s(x, y) = -\frac{1}{2} \int_0^{2\pi} p_{\text{curve}}(x, \beta) |\sin(\alpha_\beta + \beta)| d\beta.
\]
It is easy to verify that for any \( x \) inside the circular locus, the weight function satisfies\(^6,7,11,12\)
\[
2 w(x, \beta) + 2 w(- \alpha_\beta + \pi + 2 \alpha_\beta)
= \frac{ST}{2 R \cos \alpha_\beta} + \frac{S''T}{2 R \cos \alpha_\beta} = 1.
\]
A complete form of the weight function \( 2 w(x, \beta) \) with applications can be seen in Ref. 9. Dealing with \( b_s(x, y) \) in the same way, we complete the deduction from the parallel-beam formula [Eq. (3)] to the fan-beam form [Eq. (4)].

Another equivalent method to obtain Eq. (7) is to write
\[
b_s(x, y) = -\int_{\theta=0}^{\theta=2\pi} \int_{-\infty}^{\infty} W(s, \theta) p(s, \theta) \delta(s - x \cdot 0)|\sin \theta| ds d\theta,
\]
and use the coordinate change \( ds d\theta = R \cos \alpha \beta \alpha \).

We acknowledge that based on the method employed in Ref. 7, or the method employed in Ref. 6 with the new point spread functions \(-\frac{1}{\gamma} \sin \phi \) and \( \frac{1}{\gamma} \cos(\phi \mod \pi) \) \([\phi, \theta) \) the polar coordinate of \((x, y)\), alternative deductions from Eq. (3) to Eq. (4) can be constructed as well, but formula (7) generated in the above deduction can deal with both the full and short scans. In the full scan case, since Eqs. (8) and (10) may have different noise property and image qualities, one may choose the best, depending on his applications and requirements.

What is more, with the general relationship between \( d\theta \) and \( db \) in Ref. 10,
\[
d\theta = \frac{R(\beta) \cos \alpha_\beta - R'(\beta) \sin \alpha_\beta}{L} d\beta,
\]
the formula \( b_s(x, y) \) of an object inside a smooth closed convex locus can be expressed as
\[
b_s(x, y) = -\int_{\beta=0}^{\beta=2\pi} w(\alpha_\beta, \beta) p_{\text{curve}}(\alpha_\beta, \beta) |\sin(\alpha_\beta + \beta)|
\times \frac{R(\beta) \cos \alpha_\beta - R'(\beta) \sin \alpha_\beta}{L} d\beta.
\]
by repeating the above discussion, where \( R(\beta) \) describes the radius change of the scanning locus, \( w(\alpha_\beta, \beta) \) needs to be normalized for two opposite rays.

Finally, we point out that the formulation here is also valid for the classical backprojection algorithm.\(^6,7,11,12\) By removing “-” and the sinusoidal factor, Eqs. (7) and (13) become the general formulas of the classical backprojection algorithm for circular and general loci, and Eqs. (10) and (8) become Gullberg's full-scan formula and its alternative form.\(^11\) By removing “-” and the sinusoidal factor and letting the weight function in Eq. (13) cancel out the fraction factor, we arrive at Zeng-Gullberg's formula in case of the varying focal length collimator in Ref. 7 and the normality of this weight function is equivalent to Eq. (18) in Ref. 7.

By the way, there are three minor typos in Ref. 2: (1) on pages 1 and 2, two \( \int_0^{\pi} \) integrals seem \( \int_0^{2\pi} \); (2) in Eq. (12), mod \( 2\pi \) seems mod \( \pi \); and (3) in Eq. (14) cos seems cot.
ACKNOWLEDGMENTS

The author thanks an anonymous reviewer for providing Ref. 8 and Professor G. L. Zeng for providing an alternative deduction and valuable discussions.

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\(^9\) Y. Wei and G. Wang, “New relations between the divergent beam projection and Radon Transform with applications,” Inverse Problems (under review).

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