Grangeat-Type and Katsevich-Type Algorithms for Cone-Beam CT*

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Abstract: Cone-beam image reconstruction algorithms are in rapid development for major biomedical and industrial applications. In this report, we primarily focus on those algorithms that allow exact and efficient reconstruction and have potential for dynamic studies, which are recently developed Grangeat-type and Katsevich-type algorithms. This preference is due to the needs for quantitative and functional CT/micro-CT applications. Last year, Lee and Wang developed Grangeat-type half-scan cone-beam algorithms in the circular and helical scanning cases to solve the short object problem. In both the circular and helical half-scan cases, the boundaries between regions of different sample redundancies are first determined. Then, corresponding weighting functions are formulated for evaluation at various characteristic points. It was demonstrated that the Grangeat-type half-scan reconstruction clearly outperformed the Feldkamp-type half-scan reconstruction in terms of intensity dropping artifacts. In 2001, Katsevich derived the first theoretically exact reconstruction formula for the spiral cone-beam geometry in the filtered backprojection format. The limitations of this formula include a large detector window and a small radius of the object to be reconstructed. Last year, Katsevich improved his first formula remarkably. The new formula imposes little restriction on the size of the patient, and assumes a smaller detector array than the old formula. Recently, Katsevich generalized his method from the spiral scanning case to other trajectories, and proved that the earlier two formulas are special cases of his general formula. There are important needs to improve current Grangeat-type and Katsevich-type algorithms for dynamic volumetric imaging in the long object cone-beam geometry.

Key words: CT, micro-CT, cone-beam, helical/spiral scanning, half-scan, image reconstruction.

*Received on Feb. 9, 2003. This work was supported by the NIH grants R01 DC03590 and R21/33 EB001685) (本研究工作获得美国国家卫生署基金的资助)
Katsevich 推导了第一个理论上精确的螺旋锥束滤波反投影型重建公式。其局限性是探测器窗口较大和待重建物体的半径较小。2002年Katsevich改进了他的第一个公式，新的公式对病人的尺寸没有多少限制，而且相对旧公式假设了较小的探测器阵列。最近，Katsevich将他的方法推广到一般的扫描轨迹，证明了早期的两个公式是他的一般结果的特例。针对长物体的动态体成像，我们极其需要改进现有的 Grangeat 类和 Katsevich 类的算法。

关键词: CT, 显微 CT, 锥束, 螺旋扫描, 半扫描, 图像重建。

1 Introduction

Closely correlated to the development of X-ray CT, the research for high scanning speed and large volume coverage has been pursued for important biomedical and industrial applications. A famous system is the Dynamic Spatial Reconstructor (DSR) built at the Mayo Clinic in 1979[13]. In a 1991 SPIE conference, for the first time we conceptualized a spiral cone-beam scanning mode to solve the long object problem (i.e., tomographic reconstruction of a long object from longitudinally truncated cone-beam data) and published a generalized Feldkamp cone-beam image reconstruction algorithm [4]. In 1990s, single-slice spiral CT became the standard scanning mode[3,8]. In 1998, multi-slice spiral CT was introduced and has had a great impact on the field[29]. With the fast evolution of the technology, the spiral cone-beam CT scanner is emerging as the next generation workhorse[10,11].

Since several years ago, there have been increasingly stronger needs for imaging small animals, especially genetically engineered mice[12,15]. In the 1990s, a number of micro-CT systems were constructed. Most of these systems employ CCD cameras, micro-focus X-ray tubes, and have image resolutions between 20–100 μm. In recent prototypes of micro-CT systems, the data acquisition system rotates about an animal table, while in earlier systems an animal stage is rotated in a fixed data acquisition system. Such micro-imaging scanners may produce down to a few micron resolution images from cone-beam projection data[12,13].

The cone-beam reconstruction algorithms may be categorized into three types: (1) approximate, (2) exact, and (3) iterative methods. The most famous approximate image reconstruction algorithm is the Feldkamp algorithm[16]. This approach is based on a heuristic approximation to extend the circular scanning fan-beam image reconstruction formula into the circular cone-beam geometry. We extended the Feldkamp algorithm into several variants, covering spiral scanning and half-scan cases[4,17,18]. The exact image reconstruction from cone-beam data is feasible under the completeness or extended completeness condition[19,22]. Basically, it is required that any plane cutting an object to be reconstructed should contain a fan-beam or a seamless fan-beam mosaic, which contains the intersection of the plane and the object. The most popular exact cone-beam reconstruction formula is due to Grangeat, which is only for the circular scanning and spherical object case[21]. When an object to be reconstructed is long, such as a patient, longitudinal truncation of cone-beam data cannot be avoided. To solve this long object problem, the Grangeat formula has been generalized to use truncated cone-beam data[22,25]. Recently, Katsevich formulated a filtered backprojection algorithms for exact cone-beam reconstruction[26-30]. The iterative image reconstruction algorithms have been greatly refined over past years[33-38], but they are significantly slower than the non-iterative methods and need more powerful and sophisticated parallel programming techniques for practical applications. Representative cone-beam algorithms are summarized in Table 1. In this report, we primarily describe and discuss two kinds of state-of-the-art cone-beam CT algorithms, which are Grangeat-type and Katsevich-type algorithms respectively.
2 Grangeat-Type Algorithms

Because the half-scan mode shortens the data acquisition time, half-scan CT algorithms are widely used in fan-beam and cone-beam geometry. While earlier half-scan cone-beam algorithms are in the Feldkamp framework [18, 47], over the past year Lee and Wang developed Grangeat-type half-scan cone-beam algorithms in the circular and helical scanning cases to solve the short object problem [42, 45, 48]. While the Grangeat-type framework promises exact reconstruction when the scanning geometry is complete, it allows approximate reconstruction as well after missing data are appropriately estimated. Our motivation is to utilize the explicit Radon space information available in the Grangeat-type reconstruction, perform appropriate data filling in the shadow zone of the Radon space, and suppress the intensity dropping artifacts associated with the Feldkamp-type reconstruction.

As shown in Fig. 1, in the circular half-scan case without any data truncation Lee and Wang modified the original Grangeat formula into the following half-scan version [42]:

$$\frac{\partial}{\partial \rho} Rf(\rho \bar{n}) = \sum_{\lambda=1}^{2} \omega_{\lambda}(\rho \bar{n}) \frac{1}{\cos^{2} \beta} \frac{\partial}{\partial s} \int_{SO} Xf(s(\rho \bar{n}), t, \psi_{\lambda}(\rho \bar{n})) dt,$$

where

$$\psi_{1}(\rho, \theta, \varphi) = \frac{\rho}{SO \sin \theta} \quad \text{and} \quad \psi_{2}(\rho, \theta, \varphi) = \varphi + \pi - \sin^{-1} \left[ \frac{\rho}{SO \sin \theta} \right]$$

depending on a characteristic point in the Radon space.
The scanning angle \( \psi \) varies from 0 to \( \pi + 2\gamma_m \), where \( \gamma_m \) is the cone angle. In the circular full-scan case, for any characteristic point not in the shadow zone or on its surface there exist a pair of detector planes specified by the above two angle functions. However, in the circular half-scan case, such dual planes are not always available. When the dual planes are found, we are in a doubly sampled zone. When one of them is missing due to the half-scan, we are in a singly sampled zone. Relative to the full-scan, the shadow zone is increased due to a decreased amount of data from the half-scan. We formulated the boundaries and designed a smooth weighting scheme.

Then, Lee and Wang extended their results from the circular half-scan case to the case of a helical half-scan without any data truncation\[45\]. While we can calculate the line integration point analytically in a circular case, we can only do it numerically in a helical case. As a result, the Radon data can be computed in the following linear fashion:

\[
\frac{\partial}{\partial \rho} R_f (\rho \Pi) = \sum_{g=1}^{4} \omega_g R_f (\rho \Pi),
\]

where \( \omega_g \) denotes the weighting functions, \( g \) is a group identifier associated with the type of the region on the meridian plane. This form is basically the same as that with a circular half-scan but three weighting functions are needed for evaluation at a characteristic point because of existence of the triply sampled zone, while we only need two weighting functions previously.

In both the circular and helical half-scan cases, we used the zero-padding and linear interpolation methods respectively to estimate missing data in the shadow zone. The Grangeat reconstruction after zero padding in the circular case is theoretically equivalent to the Feldkamp reconstruction. The heuristics behind our choice of the linear interpolation method is that the derivative Radon data of many geometrically regular objects, such as ellipsoids or tetrahedrons, are linear along the \( \rho \) direction\[49\]. Therefore, the linear interpolation method may help effectively recover missing data, especially in some industrial CT applications. As shown in Fig. 2, it was demonstrated in numerical simulation that our Grangeat-type half-scan reconstruction clearly outperformed the Feldkamp-type half-scan reconstruction in terms of intensity dropping artifacts\[50\].

We have noticed that Noo and Heuscher independently published a Grangeat-type half-scan cone-beam reconstruction algorithm in an SPIE conference\[61\]. However, their work is only for a circular scanning locus.
and is in the filtered backprojection framework, while ours is applicable to both circular and helical scanning loci, and is in the rebinning framework. It seems that the data-filling mechanism is more flexible in our framework. Noo and Heuscher suggested that the parallel-beam approximation of cone-beam projection data be used to estimate missing data, which is done in the spatial domain. This kind of spatial domain processing is also allowed in our framework. In addition to the spatial domain approximation, the Radon domain based estimation, such as linear interpolation, and even knowledge-based interpolation, can be conveniently done in our framework. Clearly, a systematic comparison of the two algorithms is worth of further investigation.

![Figure 2. Grangeat-type helical half-scan reconstruction of the 3D Shepp-Logan phantom. (a) Original sections of the 3D Shepp-Logan phantom. (b) Feldkamp-type half-scan reconstructions, and (c) Grangeat-type half-scan reconstructions. First row: vertical slice at \( y = 0.242 \) with a circular half-scan. Second row: vertical slice at \( x = 0.363 \) with a helical half-scan (Contrast range: 1.005-1.06).]

3 Katsevich-Type Algorithms

In 2001, Katsevich derived the first theoretically exact reconstruction formula for the spiral cone beam geometry in the filtered backprojection form \[^32\]. Although its derivation is mathematically overwhelming to a general reader, the formula can be numerically implemented in three simple steps: (1) \(^1\) derivative operation of cone beam data with respect to the scanning angular parameter, (2) 1D spatially invariant filtration upon the first derivatives of cone beam data, and (3) 3D filtered backprojection to reconstruct an image. There are mainly two drawbacks with this novel formula. First, it demands a detector window substantially wider than the Tam-Danielsson requirement (Fig. 3)\[^22,40\]. Second, it requires that the radius of the object to be reconstructed is significantly smaller than the radius of the scanning locus. Last year, Katsevich improved his first formula remarkably\[^59\]. The new formula is still theoretically exact and numerically efficient while it imposes little restriction on the size of the patient, and assumes a smaller detector array than the old formula. Additionally, the new formula is approximately two times faster than the old one.

The work by Katsevich is based on the Tam-Danielsson detection geometry. When a cone-beam emits from
$y(s_0)$, the upper and lower helical paths are projected on the detector plane, as shown in Fig. 3. These projected convex/concave curves define the boundaries of Tam-Danielsson detection window, which ensures non-redundant and complete data acquisition. Any voxel inside the object is on one and only one so-called PI-line, which is spanned by two points on a spiral turn. In the perspective of that voxel, it is irradiated over an angular range of $\pi$. Interestingly, this data acquisition geometry eliminates any correlation between reconstruction at a voxel and cone-beam projections collected outside the spiral segment associated with the PI-line.[29]

![Figure 3. Tam-Danielsson detection window, which ensures non-redundant and complete data acquisition.](image)

As illustrated in Fig. 4, the main contribution by Katsevich can be summarized in the following theorem.[29]

**Katsevich Theorem:** For $f \in C_0^\infty(U)$,

$$f(x) = -\frac{1}{2\pi} \int_{I_{pt}(x)} \frac{1}{|x - y(s)|} \left[ \int_0^{2\pi} D_f(y(q), \Theta(s, x, y)) \sin \gamma \frac{dy}{\sin \gamma} \right] ds,$$

where $e(s, x) = \beta(s, x) \times u(s, x)$ and $\Theta(s, x, y) = \cos \gamma \beta(s, x) + \sin \gamma e(s, x)$.

Computationally speaking, for a given $x$, its unique PI line defined by endpoints $s_0(x)$ and $s_1(x)$ is found to delineate the integration interval $I_{pt}(x)$. For cone-beam data collected from each source position $s \in I_{pt}(x)$, the first derivative operation is performed with respect to the scanning angle parameterized by $s$.

Note that in this differentiation process, $\Theta(s, x, y)$ of involved adjacent cone-beam projections must be exactly aligned. Then, one-dimensional filtering is performed with respect to the angle $\gamma$ at every source position $s \in I_{pt}(x)$. The filtering path is the intersection of the detector surface with the plane through $x$, $y(s_0)$, $y(s_1)$ and $y(s_2)$, where $y(s_1)$, $y(s_2)$ are uniquely determined by a pre-specified function $\varphi$. The filtered data at different source positions within the PI interval contribute to the voxel reconstruction according to weights inversely proportional to the distance from $x$ to the source. In other words, a weighted backprojection is needed after the filtration to reconstruct that voxel. Some filtering paths are shown in Fig. 5. It can be observed that the detector area required in the Katsevich-type reconstruction is larger than the Tam-Danielsson window depicted by $\Gamma_{top}$ and $\Gamma_{bot}$.
\[ \phi(s, x, \gamma) = \cos \gamma \beta(s, x) + \sin \gamma \theta(s, x) \]

Figure 4. Katsevich scheme for exact and efficient reconstruction in the spiral scanning case.

\[ \int_0^{2\pi} \frac{\partial}{\partial \eta} D_f(y(q), \Theta(s, x, \gamma)) \bigg|_{\eta=0} \frac{dy}{\sin \gamma} = \frac{-|x - y(s)|}{4\pi} \int_R \tilde{f}(\xi) \tilde{B}(x, \xi) e^{-i2\pi \xi \cdot (x - y(s))} sgn(\xi \cdot e(s, x)) d\xi, \]

and show that the reconstruction formula is equivalent to the inverse Fourier transform of that function. This is equivalent to prove that \[ f(x) = \frac{1}{(2\pi)^2} \int_R \tilde{f}(\xi) B(x, \xi) d\xi \] is the inverse Fourier transform. Then, the issue becomes to prove if \[ B(x, \xi) = \sum_{s_j \in F(x)} sgn(\xi \cdot y(s_j)) sgn(\xi \cdot e(s_j, x)) = 1, \] where multiple 3D vectors...
associated the scanning locus and the detection geometry are involved. By projecting the complicated 3D relationships into 2D counterparts, we have

\[ \text{sgn}(x \cdot y(s_j)) \text{sgn}(\hat{x} \cdot e(s_j, x)) = \text{sgn}(\hat{x} \cdot y(s_j)) \text{sgn}(\hat{x} \cdot \hat{e}(s_j, x)). \]

Finally, we can prove the correctness of the Katsevich formula by exclusively evaluating all the possible geometric configurations of the vectors.

Studies on the numerical implementation of Katsevich-type formulas were performed for practical use \cite{26, 29}. A mathematically equivalent but numerically different Katsevich formula was derived to avoid numerical derivative operations among several detector planes. This memory-efficient and numerically robust formula consists of five terms, the effect of each of which on the reconstructed image was systematically evaluated \cite{29}. It turned out that one of the terms that requires computationally heavy 3D backprojection may be ignored without any serious image degradation. This approximation improved the reconstruction speed by a factor of two. A generalization of the Katsevich method from the spiral scanning case to other trajectories was described in a recent paper \cite{27}. The derivation was explicitly based on the Grangeat formula and the 3D Radon transform. It was proved that the earlier two Katsevich formulas for spiral cone beam CT are special cases of the new general formula.

4 Discussions and Conclusion

Table 2 summarizes the Grangeat-type and Katsevich-type algorithms reviewed above, along with some earlier developed representative algorithms, in reference to several most important features needed for dynamic volumetric reconstruction. As highlighted by "X" in Table 2, the current capabilities for dynamic volumetric reconstruction using cone-beam CT and micro-CT are seriously limited. There are critical and immediate needs for accurate and efficient cone-beam algorithms with high spatial, contrast and temporal resolution to support challenging innovative investigations in the long object cone-beam geometry. Cardiac and contrast imaging studies are two prominent examples. The traditional CT applications are to image the pericardium, thoracic,

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<tr>
<th>Existing Algorithms</th>
<th>TR</th>
<th>TC</th>
<th>IA</th>
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<th>CE</th>
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<tr>
<td>Feldkamp (Feldkamp et al., 84) \cite{16}</td>
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<tr>
<td>Circular half-scan Feldkamp (Gullberg &amp; Zeng, 92) \cite{39}</td>
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<td>Spiral Feldkamp (Wang et al., 91) \cite{4}</td>
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<td>PL-line based (Danielsson, 97) \cite{40}</td>
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<td>Non-tangential (Toy, 83) \cite{19}</td>
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<td>Radon space (Grangeat, 91) \cite{21}</td>
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<tr>
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<td>Mosaic Radon (Tam et al., 98) \cite{22}</td>
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<td>Quasi-exact FB (Kudo et al., 00) \cite{44}</td>
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<td>Zero boundaries (DeFrese et al., 00) \cite{25}</td>
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<td>Helical Half-scan Grangeat (Lee &amp; Wang, 03) \cite{42, 45}</td>
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Table 2. Problems of the current analytic cone-beam algorithms for dynamic volumetric reconstruction (TR: temporal resolution; TC: temporal consistency; IA: image artifacts; LR: long object reconstruction; CE: computational efficiency; "X" indicates the problematic area).
and abdominal aorta, but the closer the field of view is to the heart, the poorer the image quality becomes, because of the heart beating. Most demanding cardiac imaging used to be done by e-beam CT, which is expensive and rarely available. The arteries are too elusive to be clearly captured, since the relative rest phase of the coronary arteries is only about 60 milliseconds. Contrast CT/micro-CT studies also require high temporal resolution and temporal consistency to reveal physiological and pathological data in 3D for subsequent modeling and analysis.

We believe that the two types of the algorithms reviewed here have the potential to address this kind of biomedical needs but they should be improved by overcoming a few shortcomings. A major problem with the current Grangeat-type algorithms is the inability to reconstruct a volume of interest of a long object (patient or mouse) from a half-scan dataset. We are extending our results to address this issue. A significant drawback of the current Katsevich-type algorithms is the lack of temporal consistency (the temporal consistency requires that every voxel of the same volume be reconstructed at the same time instant) for reconstruction of time-varying structures. The above Katsevich-type algorithms perform voxel-driven reconstruction from an associated PI-line scan. As a result, each voxel is reconstructed at a time instant determined by the corresponding PI-line. Hence, the whole volume reconstructed in this way may be temporally modulated substantially, distorting the true time-varying features. Currently, we are formulating an n-PI version of the original Katsevich formula for dynamic volumetric reconstruction for greatly improved temporal consistency.

In conclusion, we have reviewed some latest results on exact and efficient cone-beam reconstruction, and pointed out some major problems with current cone-beam CT and micro-CT. These problems represent major challenges and at the same time outstanding research opportunities. We are committed to making progress along these directions, and welcome possible collaboration with interested peers.

Acknowledgment

We would like to thank Dr. Heuscher for mailing us their SPIE paper. This work was supported in part by the NIH grants R01 DC03590, R21/33 EB001685 and an internal grant for micro-CT development.

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