An Intuitive Discussion on the Ideal Ramp Filter in Computed Tomography (I)

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Abstract—In X-ray computed tomography (CT), the ideal ramp filter is a generalized function defined by the inverse Fourier transform. Similar to Dirac's discussion on the delta function, we present an intuitive discussion on the ideal ramp filter. With this concise discussion, one obtains a better understanding of the filter backprojection algorithm (FBP) and can easily construct new practical filters. © 2004 Elsevier Ltd. All rights reserved.

Keywords—X-ray CT, Image reconstruction, Ideal ramp filter, Generalized function, Filter design.

1. INTRODUCTION

X-ray computed tomography (CT) has undergone tremendous advancement over the last few years. The most popular approach for image reconstruction remains the filtered back-projection (FBP) [1,2] because of its computational advantages. Theoretically, each parallel-beam projection, \( p(t, \theta) \), is convolved with the ideal ramp filter, \( h(t) \), to obtain a filtered projection,

\[
\tilde{p}(t, \theta) = p(t, \theta) \ast h(t),
\]

where \( h(t) \) is a generalized function defined by the inverse Fourier transform,

\[
h(t) = \int_{-\infty}^{\infty} |\omega| \exp(i2\pi \omega t) \, d\omega.
\]

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In literature, there coexist two inconsistent ways to deal with generalized functions; either mathematically nonrigorous but intuitive, or mathematically rigorous but abstruse [3]. For instance, in dealing with the Dirac delta function, physicists [4] and engineers [5] treat it as a point charge at the origin or an impulse at a reference moment, denoted by

$$\delta(x) = \begin{cases} 
0, & x \neq 0, \\
\infty, & x = 0,
\end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x) \, dx = 1.$$  \hspace{1cm} (3)

Mathematicians, on the other hand, think it is logically unacceptable that a function is zero almost everywhere but has area of unity. Therefore, they rigorously define Dirac delta as a continuous linear functional mapping an infinitely differentiable compact support function $\varphi(x)$ to the number $\varphi(0)$ [3,6–8], denoted by

$$\langle \delta, \varphi \rangle = \varphi(0).$$

Certainly we have no reason to reject the mathematical rigorousness, but we really appreciate the intuitive definition and properties of Dirac delta, which brings us a "much better impression" [3, Line 13, p. 11] and enables us to easily solve real problems in many fields, even in theoretical physics [4].

Although the theory on the singular generalized functions can be found in many mathematical monographs [3,6–8], CT engineers wish an intuitive discussion on the ideal ramp filter to better understand CT algorithm and design various filters. In this paper, we try to give a self-contained intuitive discussion on the expression and properties of the ideal ramp filter and point out how to reconstruct new practical filters. Elementary calculus is enough to understand this paper. Readers interested in mathematical rigorousness are referred to suitable reference at the end of the paper.

2. AN INTUITIVE EXPRESSION OF IDEAL RAMP FILTER

Similar to Dirac’s definition for delta function, in this section, we will derive an intuitive definition for the ideal ramp filter, which is the base for the following sections.

![Figure 1. Illustration of (a) $\alpha(x)$, and (b) its derivative function $\beta(x)$.

\begin{align*}
\alpha(x) & \quad (a) \\
\beta(x) & \quad (b)
\end{align*}
Let us first consider the following function pair (see Figure 1),

\[ \alpha(x) = \begin{cases} 
\frac{1}{x}, & x \neq 0, \\
0, & x = 0, 
\end{cases} \quad (6) \]

and

\[ \beta(x) = \alpha'(x) = \begin{cases} 
-\frac{1}{x^2}, & x \neq 0, \\
+\infty, & x = 0, 
\end{cases} \quad (7) \]

where \( \beta(0) = +\infty \) is associated with the jump \( (-\infty \rightarrow +\infty) \) of \( \alpha(x) \) at the origin and should be sufficiently large that

\[ \int_{-\infty}^{+\infty} \beta(x) \, dx = [\alpha(x)]_{-\infty}^{+\infty} = 0. \quad (8) \]

Alternatively, the infinity in equation (7) can be expressed symbolically by means of a delta function,

\[ \beta(x) = \begin{cases} 
-\frac{1}{x^2}, & x \neq 0, \\
\lim_{\lambda \to 0} \frac{2\delta(x)}{|x|}, & x = 0. 
\end{cases} \quad (9) \]

Here, we recall that the similar symbol appears in the expression of delta function in polar coordinates [9].

A formal deduction of (9) is given as follows,

\[ \alpha'(x) = \left( \frac{1}{|x|} \text{sgn}(x) \right)' = \left( \frac{1}{|x|} \right)' \text{sgn}(x) + \frac{1}{|x|} \text{sgn}'(x). \quad (10) \]

Note that the right-hand sides of equations (9) and (10) are equivalent. When \( x \neq 0 \), the second term in equation (10) vanishes. Since \( 1/|x| \) is an even function, the first term becomes zero within an infinite small neighborhood of \( x = 0 \).

Simply speaking, \( \beta(x) \) is the derivative of a function with an infinite jump while the delta function is the derivative of a function with an unit jump (i.e., Heaviside step function).

Now, we are ready to express the ideal ramp filter,

\[ h(t) = \int_{-\infty}^{+\infty} |\omega| \exp(i2\pi\omega t) \, d\omega \\
= \int_{-\infty}^{+\infty} \omega \text{sgn}(\omega) \exp(i2\pi\omega t) \, d\omega \\
= \frac{1}{(i2\pi)} \int_{-\infty}^{+\infty} \text{sgn} (\omega) \exp(i2\pi\omega t) \, d\omega \\
= \frac{1}{i\pi} \int \frac{d}{dt} \left( \frac{-1}{i\pi} \alpha(t) \right) \\
= \frac{1}{2\pi^2} \beta(t). \quad (11) \]

Therefore, the explicit expression for the ideal ramp filter is

\[ h(t) = \begin{cases} 
-\frac{1}{2\pi^2 t^2}, & t \neq 0, \\
+\infty, & t = 0, 
\end{cases} \quad (12) \]

\[ \int_{-\infty}^{+\infty} h(t) \, dt = 0. \quad (13) \]
Symbolically, we also write

\[ h(t) = \begin{cases} 
\frac{1}{2\pi^2 t^2}, & t \neq 0, \\
\delta(t), & t = 0.
\end{cases} \tag{14} \]

In the next section, we will see that the positive infinity of \( h(t) \) plays an important role in FBP algorithm.

3. PROPERTIES OF THE IDEAL FILTER

With the intuitive expressions, we describe four properties of the ideal filter. From (14), the first two properties are straightforward.

PROPERTY 1. \( h(t) \) is an even function, i.e., \( h(t) = h(-t) \).

PROPERTY 2. \( h(kt) = h(t)/k^2 \) for a nonzero real number \( k \).

Note that Property 2 was used in the deduction of reconstruction algorithms for fan-beam X-ray CT [1,2].

PROPERTY 3.

\[ \int_{0}^{\pi} h(\cos(\theta - \phi)) d\theta = 0, \tag{15} \]

where \( \phi \) is any real number.

PROOF. Following Property 1, we know that \( h(\cos \theta) \) has a periodicity of \( \pi \), i.e.,

\[ h(\cos(\theta + \pi)) = h(- \cos \theta) = h(\cos \theta) \cdot \]

From Property 2, we have

\[ \int_{0}^{\pi} h(\cos(\theta - \phi)) d\theta = \int_{0}^{\pi} h(\cos \theta) d\theta = \int_{0}^{\pi} \sin^2 \theta h(\cos \theta) \csc^2 \theta d\theta \\
= - \int_{0}^{\pi} h(\cot \theta) d(\cot \theta) = \int_{-\infty}^{+\infty} h(x) dx = 0. \]

See Figure 2. This completes the proof of (15).

![Figure 2. The integrals of the ideal ramp filter along the horizontal line and the half circle (r = 1) both are equal to zero. \( \int_{0}^{\pi} h(\cos \theta) d\theta = \int_{-\infty}^{+\infty} h(x) dx = 0. \)]
PROPERTY 4. The polar coordinates of any point \( \vec{r} \) on a plan can be written as \((r, \phi)\). We have

\[
\int_0^\pi h(t) \, d\theta = \delta(\vec{r}),
\]

where \( t = r \cos(\theta - \phi) \).

ANALYSIS. For any point with \( r \neq 0 \), we have

\[
\int_0^\pi h(t) \, d\theta = \int_0^\pi h(r \cos(\theta - \phi)) \, d\theta = \frac{1}{r^2} \int_0^\pi h(\cos(\theta - \phi)) \, d\theta = 0.
\]

On the origin \((t = r = 0)\), we have

\[
\int_0^\pi h(t) \, d\theta = \int_0^\pi h(r) \, d\theta = \int_0^\pi \frac{\delta(r)}{\pi r} \, d\theta = \frac{\delta(r)}{\pi r}.
\]

Recall that \( \delta(\vec{r}) = \delta(r)/\pi r \) in polar coordinates \([9]\). Property 4 follows immediately. Notice that the situation of the denominator being zero should be interpreted in the limiting sense, i.e., \( r \to 0 \).

Property 4 indicates that if the ideal ramp filter is used for filtered back-projection, the re-constructed image of an ideal point object faithfully represents the original object. Without loss of generality, suppose a unit point \( \delta(\vec{r}) \) is located at the origin. Then, the projection data of \( \delta(\vec{r}) \) is \( \delta(t) \). The filtered projection becomes \( (\delta^* h)(t) = h(t) \). After back-projection, we obtain \( \delta(\vec{r}) \) by Property 4. Geometrically, the proof of Property 4 reveals how the ramp filter works.

That is, while at any point \( \vec{r} \neq 0 \) the superposition of positive and negative values leads to zero, at the point \( \vec{r} = 0 \) the superposition of all the positive values forms the original point source, namely (16).

This clear and simple understanding of FBP is the first benefit the intuitive method brings us.

4. PRACTICAL FILTERS AS APPROXIMATE REALIZATIONS OF THE IDEAL FILTER

Because of the singularity of the generalized function, we cannot use the ideal ramp filter directly. Most practical filters are approximate realizations of the ideal ramp filter. This conclusion is easy to verify with the help of equations (12) and (13). Let us consider two examples.

The most popular filter is the band-limited ramp filter \([1]\),

\[
h_{1}(t) = \frac{W \sin(2\pi W t)}{\pi t} - \frac{\sin^2(\pi W t)}{\pi^2 t^2} = \frac{W \sin(2\pi W t)}{\pi t} + \frac{\cos(2\pi W t)}{2\pi^2 t^2} - \frac{1}{2\pi^2 t^2},
\]

where \([-W,W]\) defines the bandwidth. Using the integral formulas in \([10]\), we can show that

\[
\int_{-\infty}^{+\infty} h_{1}(t) \, dt = 0.
\]

When \( W \to \infty \), the sine and cosine terms in (20) vibrate very quickly and cancel out so they can be considered as a zero in the sense of integration. Therefore, we have

\[
\lim_{W \to \infty} h_{1}(t) = \frac{-1}{2\pi^2 t^2}, \quad t \neq 0.
\]

Hence, based on the definition formulated by (12) and (13), we conclude that

\[
\lim_{W \to \infty} h_{W}(t) = h(t).
\]
In [2], an exponentially decaying filter was given as

$$h_\varepsilon (t) = \frac{2 \left( \varepsilon^2 - (2\pi t)^2 \right)}{(\varepsilon^2 + (2\pi t)^2)^2},$$  \hspace{1cm} (23)$$

where a constant $\varepsilon$ serves as the decaying factor. Note that the coefficient 2 was omitted in the expression in [2]. Again, it can be shown using the formulas in [10] that

$$\lim_{\varepsilon \to 0} h_\varepsilon (t) = -\frac{1}{2\pi \varepsilon^2}, \quad t \neq 0, \hspace{1cm} (24)$$

$$\int_{-\infty}^{+\infty} h_\varepsilon (t) \, dt = 0. \hspace{1cm} (25)$$

That is,

$$\lim_{\varepsilon \to 0} h_\varepsilon (t) = h (t). \hspace{1cm} (26)$$

5. NEW FILTERS AND NUMERICAL SIMULATION

What is more, based on equations (12) and (13), we can construct new approximate filters in different ways. For example, the most straightforward approximation of $h(t)$ leads to

$$h_\varepsilon (t) = \begin{cases} 
-\frac{1}{2\pi \varepsilon^2}, & |t| > \varepsilon, \\
\frac{1}{2\pi \varepsilon^2}, & |t| < \varepsilon,
\end{cases} \hspace{1cm} (27)$$

where $\varepsilon$ is a sufficiently small number (see Figure 3). In this filter, a positive constant is used in the region $|t| < \varepsilon$ to approximate an infinity magnitude in an infinitely small neighborhood. The constant $h_\varepsilon(0)$ is chosen in a way, such that

$$\int_{-\infty}^{+\infty} h_\varepsilon (t) \, dt = 0. \hspace{1cm} (28)$$

Figure 3. New digital filters designed according to the explicit expression of the ideal ramp filter. (a) Constant and (b) linear approximations to the ideal ramp filter defined in $(-\varepsilon, \varepsilon)$. 

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Based on (27), equation (1) can be expressed explicitly as

\[
\hat{p}(t, \theta) = -\frac{1}{2\pi^2} \lim_{\varepsilon \to 0} \left( \int_{-\infty}^{-\varepsilon} p\left(t-t', \theta\right) \frac{dt'}{t'^2} - \frac{1}{\varepsilon^2} \int_{-\varepsilon}^{\varepsilon} p\left(t-t', \theta\right) dt' + \int_{\varepsilon}^{\infty} p\left(t-t', \theta\right) \frac{dt'}{t'^2} \right)
= -\frac{1}{2\pi^2} \int_0^{\infty} p(t-t', \theta) + p(t+t', \theta) - 2p(t, \theta) \frac{dt'}{t'^2}.
\]  

(29)

Expression (29) clearly reveals the inherent instability of CT image reconstruction because filtered projection data are generated by summation of three infinities.

However, the filter profile depicted by (27) is discontinuous at \(|t| = \varepsilon\). These discontinuities can be removed by using a piece-wise linear approximation over the region \(|t| < \varepsilon\),

\[
h_\varepsilon(t) = \begin{cases} 
-\frac{1}{2\pi^2 t^2}, & |t| > \varepsilon, \\
\frac{1}{2\pi^2 \varepsilon^2} \left(3 - \frac{4|t|}{\varepsilon}\right), & |t| < \varepsilon,
\end{cases}
\]  

(30)

Again, the value of \(h_\varepsilon(0)\) is selected to satisfy (28). Clearly, other constraints may be imposed in the filter design process. For example, we may require both the continuity and differentiability of any order.

To demonstrate the feasibility of our proposed principles for the construction of a practical filter, we set \(\varepsilon = d\), where \(d\) is the detector width in a parallel-beam projection geometry. At points \(t = nd\), the values of (30) follows,

\[
h_d(nd) = \begin{cases} 
\frac{3}{2\pi^2 d^2}, & n = 0, \\
-\frac{1}{2\pi^2 n^2 d^2}, & n = \pm 1, \pm 2, \pm 3, \ldots
\end{cases}
\]  

(31)

We notice that after discretization, the summation of all terms in (31) is negative, rather than 0. Therefore, we have to increase \(h_\varepsilon(0)\) appropriately as follows,

\[
\bar{h}_d(nd) = \begin{cases} 
\frac{1}{6d^2}, & n = 0, \\
-\frac{1}{2\pi^2 n^2 d^2}, & n = \pm 1, \pm 2, \pm 3, \ldots
\end{cases}
\]  

(32)

so that the summation of positive and negative values is exactly zero, i.e.,

\[
\sum_{n=-\infty}^{\infty} \bar{h}_d(nd) = 0.
\]  

(33)

In fact, (33) is a discrete version of (13).

It should be emphasized that the normalized filter (32) is new, relative to that in literature [1,2]. Using (32), we reconstructed a 2D Shepp-Logan phantom from parallel-beam projection data, which produced visually the same image quality as that obtained with the Ram-Lak filter defined by [2, equation 61, p. 72], as shown in Figure 4. Quantitatively, our reconstruction yielded a root-mean-square (RMS) error of 0.075, while the counterpart with the Ram-Lak filter produced a RMS error of 0.065. The RMS error is defined as follows,

\[
rms = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} (x_{i,j}' - x_{i,j})^2 / \sum_{i=1}^{N} \sum_{j=1}^{N} x_{i,j}^2},
\]  

(34)
where \( N = 512 \), \( x_{i,j} \), and \( x'_{i,j} \) are the values of the \((i,j)\) pixel of the original and reconstructed images, respectively.

We acknowledge that our new filter produced a slightly larger error than the Ram-Lak filter in this particular case, but realize that the new filter serves only as an example of many filters that may be constructed using our approach. Note also that the RMS measures are only one of many characters we are interested. We are actively working on optimal designs of reconstruction filters [11].

A continuation of this paper will come soon.

6. CONCLUSION

In this paper, we present an intuitive and concise discussion on the definition and properties of the ideal ramp filter and its properties. A concise comparison of the Dirac Delta and the ideal ramp filter is attached as an appendix. Based on this engineer-friendly discussion one can understand filter back-projection better and construct new practical filters easily. A simple numerical simulation has demonstrated the efficacy of our approach.

For readers interested in mathematical rigorosity, the rigorous form associated with equation (9) can be found in [3, p. 40]. Equation (8) can be understood with the help of [3, p. 114] or [7,8]. Readers are referred to [6] to enjoy a lengthy and pure mathematical proof of equation (16).

REFERENCES

## APPENDIX

A CONCISE COMPARISON OF THE DIRAC DELTA FUNCTION AND THE IDEAL RAMP FILTER

<table>
<thead>
<tr>
<th>Symbol</th>
<th>$\delta(x)$</th>
<th>$h(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Definition in Fourier Domain</strong></td>
<td>$\delta(x) = \int_{-\infty}^{\infty} \exp(i2\pi x\omega) , d\omega$</td>
<td>$h(t) = \int_{-\infty}^{\infty}</td>
</tr>
<tr>
<td><strong>Definition in Spatial Domain</strong></td>
<td>$\delta(x) = \begin{cases} 0, &amp; x \neq 0, \ +\infty, &amp; x = 0. \end{cases}$</td>
<td>$h(t) = \begin{cases} -\frac{1}{2\pi^2 t^2}, &amp; t \neq 0, \ +\infty, &amp; t = 0. \end{cases}$</td>
</tr>
<tr>
<td></td>
<td>$\int_{-\infty}^{+\infty} \delta(x) , dx = 1$</td>
<td>$\int_{-\infty}^{+\infty} h(t) , dt = 0$</td>
</tr>
<tr>
<td><strong>Definition as Linear Functional</strong></td>
<td>$\langle \delta, \psi \rangle = \int_{-\infty}^{+\infty} \delta(x) \psi(x) , dx = \psi(0)$</td>
<td>$\langle h, \psi \rangle = \int_{-\infty}^{+\infty} h(t) \psi(t) , dt$</td>
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<tr>
<td></td>
<td></td>
<td>$= -\frac{1}{2\pi^2} \int_{0}^{+\infty} \psi(t) + \psi(-t) - 2\psi(0) , dt$</td>
</tr>
<tr>
<td><strong>Expression as Limit of Common Functions</strong></td>
<td>$\delta(x) = \lim_{\varepsilon \to 0} \delta_{\varepsilon}(x)$</td>
<td>$h(t) = \lim_{\varepsilon \to 0} h_{\varepsilon}(t)$</td>
</tr>
<tr>
<td></td>
<td>$\delta_{\varepsilon}(x) = \begin{cases} 0, &amp; x &gt; \varepsilon \text{ or } x &lt; -\varepsilon, \ \frac{1}{2\varepsilon}, &amp; -\varepsilon \leq x \leq \varepsilon. \end{cases}$</td>
<td>$h_{\varepsilon}(t) = \begin{cases} -\frac{1}{2\varepsilon^2 t^2}, &amp; t &gt; \varepsilon \text{ or } t &lt; -\varepsilon, \ \frac{1}{2\varepsilon^2 t^2}, &amp; -\varepsilon \leq t \leq \varepsilon. \end{cases}$</td>
</tr>
<tr>
<td><strong>Band-Limited Approximation</strong></td>
<td>$\delta_{W}(\omega) = \int_{-W}^{W} \exp(i2\pi x\omega) , d\omega$</td>
<td>$h_{W}(t) = \int_{-W}^{W} \omega \exp(i2\pi t\omega) , d\omega$</td>
</tr>
<tr>
<td></td>
<td>$= \frac{\sin(2\pi W x)}{\pi x}$</td>
<td>$= W \sin(2\pi W t) - \frac{\sin^2(\pi W t)}{\pi^2 t^2}$</td>
</tr>
<tr>
<td><strong>Exponentially Decaying Approximation</strong></td>
<td>$\delta_{e}(x) = \int_{-\infty}^{\infty} \exp(i2\pi x\omega - \varepsilon</td>
<td>\omega</td>
</tr>
<tr>
<td></td>
<td>$= \frac{2e^{\varepsilon^2}}{\varepsilon^2 + (2\pi x)^2}$</td>
<td>$= \frac{2(e^{\varepsilon^2} - (2\pi)^2)}{(\varepsilon^2 + (2\pi)^2)^2}$</td>
</tr>
<tr>
<td><strong>Derivation and Integration</strong></td>
<td>$\delta(x) = H'(x)$</td>
<td>$h(t) = \frac{1}{2\pi^2} \alpha'(t)$</td>
</tr>
<tr>
<td></td>
<td>$\int_{a}^{b} \delta(x) , dx = H(x)</td>
<td>_{a}^{b}$</td>
</tr>
<tr>
<td></td>
<td>$H(x) = \begin{cases} 0, &amp; x &lt; 0, \ \frac{1}{2}, &amp; x = 0, \ 1, &amp; x &gt; 0. \end{cases}$</td>
<td>$\alpha(t) = \begin{cases} \frac{1}{2}, &amp; t \neq 0, \ 0, &amp; t = 0. \end{cases}$</td>
</tr>
<tr>
<td>$a, b$ are any real numbers, $-\infty$, or $+\infty.$</td>
<td>$a, b$ are any real numbers, $-\infty$, or $+\infty.$</td>
<td></td>
</tr>
<tr>
<td><strong>Simple Properties</strong></td>
<td>$\delta(-x) = \delta(x)$</td>
<td>$h(-t) = h(t)$</td>
</tr>
<tr>
<td></td>
<td>$\delta(kx) = \frac{1}{</td>
<td>k</td>
</tr>
<tr>
<td><strong>Fourier Series</strong></td>
<td>$\delta(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \exp(i2\pi nx)$</td>
<td>$h(t) = \sum_{n=-\infty}^{\infty} \left( \frac{2}{\pi^2} \frac{(-1)^n + 2n}{n} \sin(n\pi t) \right) \exp(i2\pi nt)$</td>
</tr>
<tr>
<td></td>
<td>$-\frac{1}{2} \leq x \leq \frac{1}{2}$</td>
<td>$\text{Si}(z) = \int_{0}^{z} \frac{\sin u , du}{u}$ $-\frac{1}{2} \leq t \leq \frac{1}{2}$</td>
</tr>
<tr>
<td>Symbol</td>
<td>( \delta(x) )</td>
<td>( h(t) )</td>
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</tbody>
</table>
| Ideal Filter for Fan Beam CT | \[
\delta(\sin x) = \sum_{n=-\infty}^{\infty} \delta(x - n\pi) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \exp(2in\pi) = \frac{1}{\pi} (1 + 2\cos 2x + 2\cos 4x + \cdots + 2\cos(2Nx) + \cdots) = \lim_{N \to 0} \frac{\sin(2N+1)x}{\pi \sin x}
\] | \[
\begin{align*}
h(\sin \gamma) &= -\frac{1}{2\pi^2} \cos^2 \gamma = \sum_{n=-\infty}^{\infty} h(\gamma - n\pi) = \frac{1}{\pi^2} \sum_{n=-\infty}^{\infty} |n| \exp(2in\gamma) \\
&= \frac{1}{\pi^2} (2\cos 2\gamma + 4\cos 4\gamma + 6\cos 6\gamma + \cdots + 2N \cos(2N\gamma) + \cdots) = \frac{1}{\pi^2} \lim_{N \to \infty} \left( \frac{(N+1) \sin(2N+1)\gamma}{\sin \gamma} - \frac{\sin^2(N+1)\gamma}{\sin^2 \gamma} \right)
\end{align*}
\] |
| Important Relation | \[
\int_{0}^{\pi} h(t) \, d\theta = \delta(\theta), \text{ where } \theta \text{ is a point in the plane, } (r, \phi) \text{ is the polar coordinate of the point, and } t = r \cos(\theta - \phi).
\] |